Notes on LTI Systems

Version 1.1

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These notes were developed for use in a beginning graduate course in the Department of Electrical and Computer Engineering, Johns Hopkins University, over the period 2000 – 2005. They currently are in draft form, and they may be subject to further development in the future.

As indicated by the *Table of Contents*, the notes cover basic representations, system properties, and properties of feedback. The treatment is very much theory-oriented, but the focus is on single-input, single-output systems and elementary mathematical techniques.

Prerequisites for the material are a typical undergraduate course in control systems, though a basic introduction to signals and systems might suffice.

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0. Introduction

In many areas of engineering and science, linear time-invariant systems, known hereafter as *LTI systems*, are used as models of physical processes. In undergraduate courses the typical student has encountered a number of different representations for LTI systems. Described for the case of a unilateral, scalar input and output signals, $u(t)$ and $y(t)$, signals that are defined for $t \geq 0$, these representations include the

(i) *convolution* representation:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau$$

where the *unit-impulse response* $h(t)$ is a real function that provides a description of the system.

(ii) *transfer function* representation:

$$Y(s) = H(s)U(s)$$

where $Y(s)$, $H(s)$, and $U(s)$ are, respectively, the unilateral Laplace transforms of $y(t)$, $h(t)$, and $u(t)$. Here $H(s)$, a complex-valued function of the complex variable s, called the *transfer function*, provides a description of the system.

(iii) n^{th} *-order differential equation* representation:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = b_n u^{(n)}(t) + b_{n-1}u^{(n-1)}(t) + \cdots + b_1 u^{(1)}(t) + b_0 u(t)$$

where the parenthetical superscripts indicate time derivatives, and the real coefficients a_k and b_k describe the system.

(iv) n*-dimensional state equation* representation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t)$ is the $n \times 1$ *state vector,* and the coefficient matrices describe the system.

Each of these representations has utility for specific issues or analysis/design objectives, and each has an attendant set of assumptions, as well as advantages and disadvantages. One observation that can be made at the outset is that the first two representations involve only the input and output signals, $u(t)$ and $y(t)$, along with a signal (or Laplace transform) that represents the system. Furthermore, *initial conditions* are implicitly assumed to be zero, since an identically-zero input signal yields an identically-zero output signal. The n^{th}-order differential equation representation involves the input and output signals, and also their time derivatives. In addition the possibility of nonzero initial conditions is familiar from courses on differential equations. The n-dimensional state equation introduces new variables, the n components of the state vector $x(t)$, that are involved in relating the input signal to the output signal. In this sense it is rather different from the representations (i) – (iii). The state equation representation also admits the possibility of nonzero initial conditions on $x(t)$.

Remark
It is interesting to observe how the most elementary LTI system, the identity system, where the output signal is identical to the input signal, fits within these representations. For the convolution

representation, we are led to choose $h(t) = \delta(t)$, the *unit-impulse* function, for the usual sifting property then verifies that for any continuous signal $u(t)$ the response is $y(t) = u(t)$. For input signals that are not continuous functions, technical issues arise. To give an extreme case, we would need to rely on the technically questionable convolution of two impulses to verify that the response of the identity system to a unit-impulse input is indeed a unit impulse. The transfer function of the identity system must of course be unity, while the differential equation representation of the identity system reduces more-or-less transparently to the $n = 0$ case with $a_0 = b_0 = 1$. A state equation representation of the identity system would involve taking $C = 0$ and $D = 1$, with dimension n having any nonnegative integer value, though taking $n = 0$ seems preferable on grounds of economy. □

Despite this diversity of representations for LTI systems, matters are well in hand in that the relationships among them are well understood, and concepts or results in terms of one representation usually can be interpreted in terms of another. But not always; for example, there are state equation descriptions that cannot be rendered into an n^{th}-order differential equations.

One of our objectives is to present these relationships in a more careful and complete way than is typical in undergraduate courses. Another is to elucidate the relative advantages of one representation compared to another when addressing certain types of issues. Overall, our treatment should provide a useful bridge from the typical undergraduate control course to more advanced graduate courses in the analysis and design of multi-input, multi-output LTI systems, linear systems that are not time invariant, and nonlinear systems.

It turns out that the convolution representation is in many ways the most general representation of the four. For example, the LTI system with unit-impulse response $h(t) = e^{t^2}$, $t \geq 0$, cannot be represented using the other options. However, from the viewpoint of many applications, the state equation is the most basic, and this is where we start. In the process of developing relationships among the four representations, issues of comparative generality will become clearer. Also we should note that every approach, technique, tool, and trick used in the development can be refined, or generalized, or rejected in favor of alternatives. Our choices are based largely on the objective of technical simplicity rather than mathematical elegance.

1. LTI Systems Described by State Equations

We consider the state equation representation

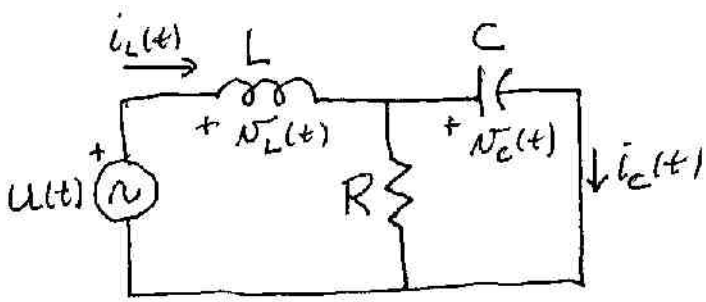
$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad t \geq 0\,, \quad x(0) = x_o$$
$$y(t) = Cx(t) + Du(t)$$

where the input signal $u(t)$ and the output signal $y(t)$ are scalar signals defined for $t \geq 0$, and the state $x(t)$ is an $n \times 1$ vector signal, the entries of which are called the *state variables*. Conformability dictates that the coefficient matrices $A, B, C,$ and D have dimensions $n \times n,\, n \times 1,\, 1 \times n,$ and 1×1, respectively. We assume that these matrices have real entries, unless otherwise noted.

For many topics the "*D*-term" in the state equation plays little role beyond that of an irritating side issue. Regardless, we retain it for most of our discussions on two grounds: it appears in very simple examples, and it seems only reasonable that the class of LTI systems we study should include the *identity* system.

We use a few simple examples to illustrate concepts throughout the treatment.

Example
To obtain a state equation description for an RLC electrical circuit, choosing capacitor voltages and inductor currents as state variables works in all but a few, special situations. The procedure is to label all capacitor and inductor voltages and currents, and then use Kirchoff's laws to derive equations of the appropriate form in terms of these state variables. Consider the circuit



with voltage source input $u(t)$, and current output $i_C(t)$ as shown. Using the labeled currents and voltages and applying KVL to the outer loop gives

$$L\frac{d}{dt}i_L(t) = u(t) - v_C(t)$$

This is in the desired form – derivative of a state variable in terms of the input signal and the state variables. Applying KCL at the top node gives

$$C\frac{d}{dt}v_C(t) = i_L(t) - \frac{1}{R}v_C(t)$$

again an expression in the form we seek. Defining the state vector as

$$x(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$$

these two scalar equations can be packed into the 2×1 vector state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} x(t) + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t)$$

The output signal in this case is

$$i_C(t) = C\frac{d}{dt}v_C(t)$$

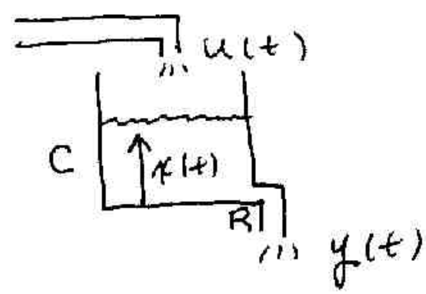
a quantity that is also directly described in terms of the state variables by the KCL result above. This can be written in the form

$$y(t) = \begin{bmatrix} 1 & -1/R \end{bmatrix} x(t)$$

to complete the state equation. The initial value $x(0)$ is of course given by the initial values of the inductor current and capacitor voltage. □

In this example a judicious choice of Kirchoff's laws led directly to equations of the appropriate form for the state-variable derivatives and for the output signal in terms of the state variables. In other cases additional circuit equations might be needed to eliminate unwanted variables and obtain the appropriate state equation format.

Example
A simple, cartoon version of hydrology models proves useful in illustrating a number of concepts. Consider a cylindrical water bucket with cross-sectional area C ft^2 and water depth $x(t)$ ft as shown below.



We assume the inflow rate is $u(t)$ ft^3/sec and the outflow rate through a hole in the bottom of the bucket is $y(t)$ ft^3/sec. The basic, and somewhat unrealistic, assumption we make is that the outflow rate is proportional to the depth of water in the bucket,

$$y(t) = \frac{1}{R}x(t)$$

where the constant R has appropriate units. This assumption leads to a linear bucket model that, in some situations to be discussed in the sequel, is reasonable, at least as an approximation.

A model for the bucket system follows directly from a calculation of the rate-of-change of the volume of water in the bucket as inflow rate minus the outflow rate:

$$C\,\dot{x}(t) = u(t) - y(t)$$

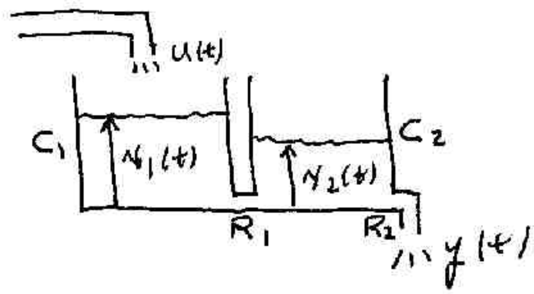
Using the outflow rate assumption, this can be rewritten in state equation form as

$$\dot{x}(t) = \frac{-1}{RC}x(t) + \frac{1}{C}u(t)$$

$$y(t) = \frac{1}{R}x(t)$$

where the state – the water depth – is a scalar. Of course there is an implicit assumption in play here: the initial state $x(0) = x_o$ must be nonnegative, and the inflow and outflow rates must be nonnegative for every $t \geq 0$. In the sequel the nonnegativity and linearity assumptions will be addressed further.

The utility of the linear bucket system as an example rests on the ability to interconnect bucket systems in various ways. Consider the parallel bucket system, two buckets with cross-sectional areas C_1 and C_2 connected via hose, with a second hole in the second bucket, and depths and input and output flow rates as labeled.



We make the linearity assumptions that

$$y(t) = \frac{1}{R_2}\, x_2(t)$$

and that the flow rate in the connecting hose is proportional to the difference in depths in the two buckets. Assuming positive flow rate from left to right in the hose, the rates of change of volume in the two buckets are given by

$$C_1\, \dot{x}_1(t) = u(t) - \frac{1}{R_1}[x_1(t) - x_2(t)]$$

$$C_2\, \dot{x}_2(t) = \frac{1}{R_1}[x_1(t) - x_2(t)] - \frac{1}{R_2}x_2(t)$$

Packing these into vector form, with $x(t)$ as the 2×1 vector with components $x_1(t)$ and $x_2(t)$ yields the state equation

$$\dot{x}(t) = \begin{bmatrix} -1/(R_1C_1) & 1/(R_1C_1) \\ 1/(R_1C_2) & -1/(R_1C_2) - 1/(R_2C_2) \end{bmatrix} x(t) + \begin{bmatrix} 1/C_1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1/R_2 \end{bmatrix} x(t)$$

The series connection of buckets is straightforward to analyze, and is given as an exercise. □

Example

Many models in classical physics appear in the form of a scalar, n^{th}-order, constant-coefficient, linear differential equation of the form

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = b_0 u(t)$$

where $y(t)$ and $u(t)$ represent the output and input signals, and initial conditions are given as

$$y(0),\, y^{(1)}(0), \ldots,\, y^{(n-1)}(0)$$

A standard, simple trick for converting this to state equation format simply involves naming the output and its first $n-1$ derivatives as the state variables:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y^{(1)}(t) \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

Then the state-variable time derivatives can be written in terms of state variables and the input signal as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= -a_0x_1(t) - a_1x_2(t) - \cdots - a_{n-1}x_n(t) + b_0u(t) \end{aligned}$$

where the first $n-1$ of these equations are matters of definition, and the last equation is obtained by writing the differential equation in terms of the new variables. Packing this collection into vector form, and writing the obvious output equation gives the state equation description

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} x(t)$$

Initial conditions on the output and its time derivatives provide an initial state via the definitions,

$$x(0) = \begin{bmatrix} y(0) \\ y^{(1)}(0) \\ \vdots \\ y^{(n-2)}(0) \\ y^{(n-1)}(0) \end{bmatrix}$$

Linearization

An important motivation for studying LTI systems is their utility in approximating nonlinear systems near a set point. First, recall Taylor approximation for a differentiable, real-valued function of n real variables, written in vector notation as $f(x)$. Given a point $\tilde{x} \in R^n$ of interest, and a small (in norm) $x_\delta \in R^n$, we can write as an approximation

$$f(\tilde{x} + x_\delta) \approx f(\tilde{x}) + \frac{\partial f}{\partial x}(\tilde{x})\, x_\delta$$

where the indicated derivative is a $1 \times n$ row vector of scalar partial derivatives,

$$\frac{\partial f}{\partial x}(\tilde{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(\tilde{x}) & \cdots & \dfrac{\partial f}{\partial x_n}(\tilde{x}) \end{bmatrix}$$

That is,

$$f(\tilde{x} + x_\delta) - f(\tilde{x}) \approx \frac{\partial f}{\partial x}(\tilde{x})\, x_\delta$$

which gives a linear approximation for the deviation in the value of the function in terms of the (assumed small) deviation of the independent variable.

It seems natural to use this idea to obtain a linear-system approximation of the behavior of a nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_o$$

where $f : R^n \times R \to R^n$ is differentiable, by invoking a linear approximation of f. Indeed this works in many situations, though we will not take the time to delineate conditions and limitations.

Suppose for some constant input signal of interest, $u(t) = \tilde{u}$, $t \geq 0$, there is a constant solution $x(t) = \tilde{x}$, $t \geq 0$ (among other things, note that $x_o = \tilde{x}$). That is,

$$0 = f(\tilde{x}, \tilde{u})$$

Next consider an input signal $u(t)$ that remains close to $\tilde{u}$ in the sense that the deviation quantity

$$u_\delta(t) = u(t) - \tilde{u}$$

remains small (in absolute value) for all $t \geq 0$, and an initial state $x(0)$ that is close to $\tilde{x}$ in the sense that the deviation vector

$$x_\delta(0) = x_o - \tilde{x}$$

is small (in norm). It might be hoped that the resulting solution $x(t)$ remains close to $\tilde{x}$ in the sense that the deviation

$$x_\delta(t) = x(t) - \tilde{x}$$

remains small (in norm) for all $t \geq 0$. In terms of these deviation quantities, we can write the nonlinear system as

$$\frac{d}{dt}\left[\tilde{x} + x_\delta(t)\right] = f(\tilde{x} + x_\delta(t), \tilde{u} + u_\delta(t)), \quad \tilde{x} + x_\delta(0) = x_o$$

That is

$$\dot{x}_\delta(t) = f(\tilde{x} + x_\delta(t), \tilde{u} + u_\delta(t)), \quad x_\delta(0) = x_o - \tilde{x}$$

Now we apply the Taylor approximation to each component of f, namely, each

$$f_i : R^n \times R \to R^n$$

to write (keeping the arguments x and u distinct)

$$f_i(\tilde{x} + x_\delta, \tilde{u} + u_\delta) \approx f_i(\tilde{x}, \tilde{u}) + \frac{\partial f_i}{\partial x}(\tilde{x}, \tilde{u}) x_\delta + \frac{\partial f_i}{\partial u}(\tilde{x}, \tilde{u}) u_\delta, \quad i = 1, \ldots, n$$

Of course, each $f_i(\tilde{x}, \tilde{u}) = 0$, $\partial f_i / \partial x$ is $1 \times n$, and $\partial f_i / \partial u$, is a scalar. Packing the approximations into vector form gives

$$f(\tilde{x} + x_\delta, \tilde{u} + u_\delta) \approx \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}) x_\delta + \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}) u_\delta$$

where the i, j entry of the $n \times n$ matrix

$$\frac{\partial f}{\partial x}(\tilde{x}, \tilde{u})$$

is

$$\frac{\partial f_i}{\partial x_j}(\tilde{x}, \tilde{u})$$

and the i^{th} entry of the $n \times 1$ matrix

$$\frac{\partial f}{\partial u}(\tilde{x},\tilde{u})$$

is

$$\frac{\partial f_i}{\partial u}(\tilde{x},\tilde{u})$$

In this way we pose the linear state equation

$$\dot{x}_\delta(t) = \frac{\partial f}{\partial x}(\tilde{x},\tilde{u})x_\delta(t) + \frac{\partial f}{\partial u}(\tilde{x},\tilde{u})u_\delta(t)\,, \quad x_\delta(0) = x_o - \tilde{x}$$

whose behavior for small initial states, $x_\delta(0)$ and small input signals $u_\delta(t)$ is hoped to approximate the behavior of the nonlinear system in the sense that

$$x(t) \approx \tilde{x} + x_\delta(t)\,, \quad t \geq 0$$

If there is a (nonlinear) output equation,

$$y(t) = h(x(t),u(t))$$

another standard Taylor approximation of the function $h(x,u)$ about $(\tilde{x},\tilde{u})$ gives

$$y_\delta(t) = \frac{\partial h}{\partial x}(\tilde{x},\tilde{u})x_\delta(t) + \frac{\partial h}{\partial u}(\tilde{x},\tilde{u})u_\delta(t)$$

as an approximate output equation, where the output deviation is defined by

$$y_\delta(t) = y(t) - h(\tilde{x},\tilde{u})$$

Example

Written in a familiar notation, the description of a pendulum with a mass m suspended by a massless rod from a frictionless pivot is

$$\ddot{y}(t) + \frac{g}{l}\sin(y(t)) = \frac{1}{ml^2}u(t)$$

where $y(t)$ is the angle of the rod measured from the vertical below the pivot, and $u(t)$ is the torque on the rod at the pivot. We can put this second-order nonlinear differential equation into state equation form using the same procedure as the linear case. Letting

$$x_1(t) = y(t)\,, \quad x_2(t) = \dot{y}(t)$$

gives

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) + \frac{1}{ml^2}u(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Taking $\tilde{u} = 0$, we see that $\tilde{x} = 0$ is one solution and the corresponding linearized state equation is specified by

$$A = \frac{\partial}{\partial x}\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) + \frac{1}{ml^2}u \end{bmatrix}\Bigg|_{x=0,u=0} = \begin{bmatrix} 0 & 1 \\ \frac{-g}{l} & 0 \end{bmatrix}$$

$$B = \frac{\partial}{\partial u}\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) + \frac{1}{ml^2}u \end{bmatrix}\Bigg|_{x=0,u=0} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

Of course,

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

This can be recognized as the state equation for a harmonic oscillator, and we are led to the notion that for small initial states and small input signals the techniques for LTI systems that we discuss will be useful for analyzing the behavior of the pendulum.

We can also consider the constant solution

$$\tilde{u} = 0\,, \quad \tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

in which case the linearized state equation is specified by

$$A = \frac{\partial}{\partial x}\left[\begin{matrix} x_2 \\ -\dfrac{g}{l}\sin(x_1) + \dfrac{1}{ml^2}u \end{matrix}\right]\Bigg|_{x=\begin{bmatrix}\pi \\ 0\end{bmatrix}, u=0} = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{l} & 0 \end{bmatrix}$$

$$B = \frac{\partial}{\partial u}\left[\begin{matrix} x_2 \\ -\dfrac{g}{l}\sin(x_1) + \dfrac{1}{ml^2}u \end{matrix}\right]\Bigg|_{x=0, u=0} = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix}$$

In this case we might well be suspicious of the approximation notion, because of the obvious behavior of the pendulum for initial states near the upright position.

Example

Consider the basic water bucket system in our first example, but suppose the outflow rate is given by a nonlinear function of water depth:

$$y(t) = \frac{1}{R}\sqrt{x(t)}$$

Consideration of the rate of change of volume now leads to the state equation

$$\dot{x}(t) = \frac{-1}{RC}\sqrt{x(t)} + \frac{1}{C}u(t)$$

$$y(t) = \frac{1}{R}\sqrt{x(t)}$$

where, again, only nonnegative values of the variables are permitted. Choosing the constant solution of zero inflow rate and zero depth, $\tilde{u} = 0$, $\tilde{x} = 0$, calculation of the linearized state equation begins innocently enough,

$$A = \frac{\partial}{\partial x}\left[\frac{-1}{RC}\sqrt{x} + \frac{1}{C}u\right]\Bigg|_{x=u=0} = \frac{-1}{2RC\sqrt{x}}\Bigg|_{x=0}$$

but ends badly! The problem, of course, is that the right-hand side of the nonlinear state equation is not differentiable with respect to x at $x = 0$.

If we choose any positive inflow rate, $\tilde{u} > 0$, it is easy to verify that there is a constant solution given by

$$\tilde{x} = R^2\tilde{u}^2$$

and, as expected, the corresponding constant outflow rate is $\tilde{y} = \tilde{u}$. The coefficients of the linearized system are given by

$$A = \frac{\partial}{\partial x}\left[\frac{-1}{RC}\sqrt{x} + \frac{1}{C}u\right]\Bigg|_{x=\tilde{x},u=\tilde{u}} = \frac{-1}{2RC\sqrt{x}}\Bigg|_{x=R^2\tilde{u}^2} = \frac{-1}{2R^2C\tilde{u}}$$

$$B = \frac{\partial}{\partial u}\left[\frac{-1}{RC}\sqrt{x} + \frac{1}{C}u\right]\Bigg|_{x=\tilde{x},u=\tilde{u}} = \frac{1}{C}$$

$$C = \frac{\partial}{\partial x}\left[\frac{1}{R}\sqrt{x}\right]\Bigg|_{x=\tilde{x},u=\tilde{u}} = \frac{1}{2R\sqrt{x}}\Bigg|_{x=R^2\tilde{u}^2} = \frac{1}{2R^2\tilde{u}}$$

$$D = \frac{\partial}{\partial u}\left[\frac{1}{R}\sqrt{x}\right]\Bigg|_{x=\tilde{x},u=\tilde{u}} = 0$$

Therefore, with

$$\begin{aligned} u_\delta(t) &= u(t) - \tilde{u} \\ x_\delta(t) &= x(t) - \tilde{x} = x(t) - R^2\tilde{u}^2 \\ y_\delta(t) &= y(t) - \tilde{u} \end{aligned}$$

we obtain the approximating linear state equation (for $\tilde{u} > 0$)

$$\begin{aligned} \dot{x}_\delta(t) &= \frac{-1}{2R^2C\tilde{u}}\,x_\delta(t) + \frac{1}{C}\,u_\delta(t)\,, \quad x_\delta(0) = x_o - R^2\tilde{u}^2 \\ y_\delta(t) &= \frac{1}{2R^2\tilde{u}}\,x_\delta(t) \end{aligned}$$

Changes of state variables

Reflection on the examples we have considered indicates that various choices can be made for the state variables in developing a state equation description for a system. For the electrical circuit, voltages at various nodes and currents in various loops could be chosen. And for the parallel bucket system, we could choose the water depth in the first bucket and the difference in depths between the first and second buckets as state variables. In the bucket case, this corresponds to considering the effect of the variable change

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

on the original state equation.

In general, for the state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \quad x(0) = x_o \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

we consider a linear change of state vector of the form

$$z(t) = P^{-1}x(t)$$

where P is an $n \times n$, invertible matrix. (We usually think of P as a real matrix, though complex variable changes will arise for mathematical purposes.) Then it is easy to see that

$$\begin{aligned} \dot{z}(t) &= P^{-1}\dot{x}(t) = P^{-1}Ax(t) + P^{-1}Bu(t) \\ &= P^{-1}APz(t) + P^{-1}Bu(t) \end{aligned}$$

and

$$y(t) = CPx(t) + Du(t)$$

The initial state in terms of the new state vector is

$$z(0) = P^{-1}x_o$$

The new state equation, in terms of $z(t)$, has the same form as the original (which is why only linear, constant variable changes are considered), and either state equation can be used to represent the system. Indeed, there are an infinite number of choices for the invertible, $n \times n$ matrix P, and so there are an infinite number of state equations that can be used to describe a given system!

In the sequel we will consider a small number of variable changes that are useful for particular mathematical purposes. In most cases the new state variables are not physically intuitive and this is the price paid for mathematical simplicity.

Distinct-Eigenvalue Diagonal Form

Suppose for the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad x(0) = x_o$$
$$y(t) = Cx(t) + Du(t)$$

the $n \times n$ matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$. Some eigenvalues may be complex, but these must occur in conjugate pairs. (By default, A has real entries, so the characteristic polynomial of A has real coefficients, and complex roots of real-coefficient polynomials occur in complex conjugate pairs.) We denote the corresponding eigenvectors by $p_1, \ldots, p_n$. That is, each p_i is a nonzero vector that satisfies

$$Ap_i = \lambda_i p_i\,, \quad i = 1, \ldots, n$$

Consider the change of state vector given by arranging the eigenvectors as the columns of a matrix:

$$z(t) = P^{-1}x(t) = \left[\, p_1 \mid p_2 \mid \quad \cdots \quad \mid p_n \,\right]^{-1} x(t)$$

Of course, we need to establish conditions under which this variable-change matrix, often called the *modal* matrix, indeed is invertible. Statements of a necessary and sufficient condition would involve notions such as the algebraic and geometric multiplicity of eigenvalues, so we state only a basic sufficient condition:

Theorem
If the eigenvalues of A are distinct, then the modal matrix is invertible.

Proof
We will assume that the eigenvalues of A are distinct and that P is not invertible, and arrive at a contradiction that establishes the result. Since P is not invertible, its columns are linearly independent, and we can assume that the first column can be written as a linear combination of the remaining columns:

$$p_1 = \alpha_2 p_2 + \cdots + \alpha_n p_n$$

Multiplying this expression on the left by $(A - \lambda_2 I)$ yields

$$Ap_1 - \lambda_2 p_1 = \alpha_2 (Ap_2 - \lambda_2 p_2) + \alpha_3 (Ap_3 - \lambda_2 p_3) + \cdots + \alpha_n (Ap_n - \lambda_2 p_n)$$

or, using the eigenvalue/eigenvector equation,

$$(\lambda_1 - \lambda_2) p_1 = \alpha_3 (\lambda_3 - \lambda_2) p_3 + \cdots \alpha_n (\lambda_n - \lambda_2) p_n$$

Multiplying this expression on the left by $(A - \lambda_3 I)$ will, in a similar fashion, remove the first term on the right and yield

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) p_1 = \alpha_4 (\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3) p_4 + \cdots \alpha_n (\lambda_n - \lambda_2)(\lambda_n - \lambda_3) p_n$$

Continuing this process leads to, after multiplication by $(A-\lambda_n I)$,

$$(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)p_1=0$$

Because the eigenvalues are distinct, this implies that $p_1=0$, which is impossible since p_1 is an eigenvector. □

It remains to compute the special form of the coefficient matrices of the new state equation, and the form of $P^{-1}AP$ involves a product of partitioned matrices, the result of which motivates the terminology:

$$\begin{aligned}
P^{-1}AP &= \left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right]^{-1} A\left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right] \\
&= \left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right]^{-1} \left[Ap_1 \,|\, Ap_2 \,|\, \cdots \,|\, Ap_n\right] \\
&= \left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right]^{-1} \left[\lambda_1 p_1 \,|\, \lambda_2 p_2 \,|\, \cdots \,|\, \lambda_n p_n\right] \\
&= \left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right]^{-1} \left[p_1 \,|\, p_2 \,|\, \cdots \,|\, p_n\right] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}
\end{aligned}$$

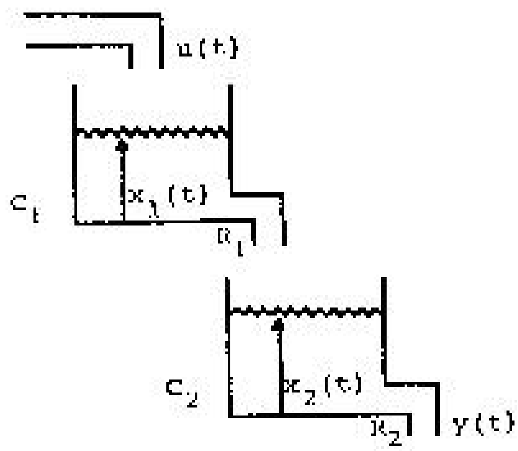
The coefficients $P^{-1}B$ and CP have no special structure, and so the advantage of the variable transformation is the diagonal "A-matrix." Of course, in the case where A has complex eigenvalues, the matrix P will have complex entries, and the new state equation will have complex coefficients. This fact obscures intuition that might have been present in the original state equation model, and is a price paid for the simple mathematical form obtained.

Remark

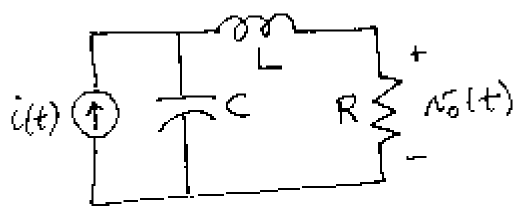
The distinct-eigenvalue hypothesis in our theorem is, as shown above, sufficient for diagonalizability, but it is not necessary. Readers interested in more general developments should investigate the *Jordan form* for matrices, a form that includes diagonal matrices as a special case.

Exercises

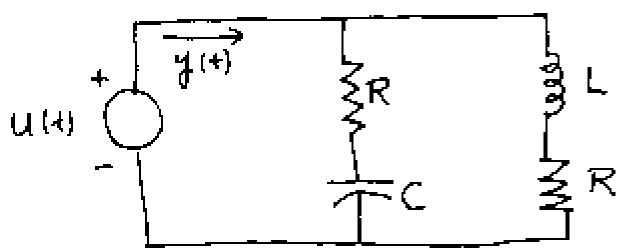
1. Using the usual assumptions write a linear state equation for the series bucket system shown below.



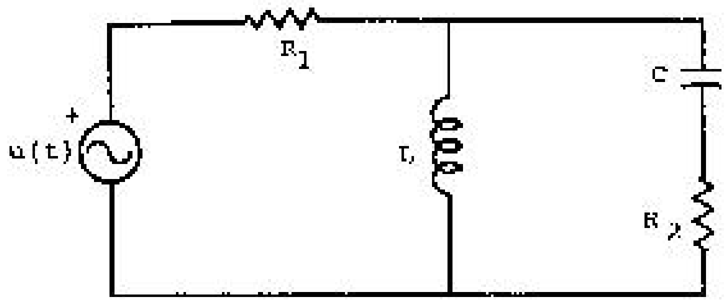
2. Write a linear state equation for the electrical circuit shown below, where the current $i(t)$ is the input and the output is the voltage $v_o(t)$.



3. Write a linear state equation for the electrical circuit shown below, where the voltage $u(t)$ is the input and the output is the current *y(t)*.



4. Write a linear state equation for the electrical circuit shown below, where the voltage $u(t)$ is the input and the output is the current through R_2.



5. Rewrite the n^{th} – order differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_0 u(t) + b_1 \dot{u}(t)$$

as a dimension-*n* linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Hint: Let $x_n(t) = y^{(n-1)}(t) - b_1 u(t)$

6. A two-input, single-output system is described by

$$\ddot{y}(t) + \sin(y(t))\,\dot{y}(t) + u_2(t)y(t) = u_1 + u_2$$

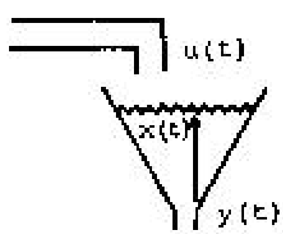
Compute the linearized state equation that describes this system about the constant operating point corresponding to $\tilde{u}_1 = 0$, $\tilde{u}_2 = 1$

7. Consider the nonlinear state equation (a so-called *bilinear* state equation)

$$\dot{x}(t) = Ax(t) + Dx(t)u(t) + Bu(t)\,, \quad x(0) = x_o$$
$$y(t) = Cx(t)$$

where *A* and *D* are $n \times n$, *B* is $n \times 1$, and *C* is $1 \times n$. What is a necessary and sufficient condition for the system to have a constant nominal solution for a constant nominal input, $u(t) = \tilde{u}$? What is the linearized state equation about such a nominal solution.

8. Consider a cone-shaped bucket depicted below, with the cone such that when $x(t) = 1$ the surface area of the water is 4. The orifice is such that $y(t) = (1/3)x(t)$. Compute a linearized state equation description about the constant operating point with $\tilde{x} = 2$. (A math handbook will remind you that a cone of height *x* and base radius *r* has volume $V = (\pi/3)r^2 x$.)



9. For the nonlinear state equation

$$\dot{x} = \begin{bmatrix} -x_2 + u \\ x_1 - 2x_2 \\ x_1 u - 2x_2 u \end{bmatrix}, \quad y = x_3$$

show that for every constant nominal input $\tilde{u}$ there is a constant nominal trajectory $\tilde{x}$. What is the constant nominal output $\tilde{y}$ in terms of $\tilde{u}$? Explain. Linearize the state equation about an arbitrary constant nominal. If $\tilde{u} = 0$ and $x_\delta(0) = 0$, what is the response $y_\delta(t)$ of the linearized state equation for any $u_\delta(t)$? (Solution of the state equation is not needed to answer this.)

10. For the nonlinear state equation

$$\dot{x}(t) = \begin{bmatrix} x_1(t) + u(t) \\ 2x_2(t) + u(t) \\ 3x_3(t) + x_1^2(t) - 4x_1(t)x_2(t) + 4x_2^2(t) \end{bmatrix}$$
$$y(t) = x_3(t)$$

compute a constant solution given any constant input $u(t) = \tilde{u}$. Linearize the state equation about such a constant operating point. If $x_\delta(0) = 0$, what is the response $y_\delta(t)$ given $u_\delta(t)$? (You don't need to solve the state equation to answer the question.)

11. Find the linearized state equation that approximately describes the behavior of the nonlinear system

$$\ddot{y}(t) + \dot{y}(t)\sin(y(t)) + (u(t) - 1)y(t) = 0$$

about the constant operating point corresponding to $u(t) = 2,\, t \geq 0$.

2. LTI State Equation Solutions

The first issue to address is existence and uniqueness of solutions, for $t \ge 0$, to the state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_o$$
$$y(t) = Cx(t) + Du(t)$$

given an initial state x_o and an input signal $u(t)$, $t \ge 0$. However, we will leave that to mathematics courses, and simply apply perhaps the most basic method of solving differential equations - solution via power series. This leads to a formula for a solution when the input is (identically) zero, and from there we discuss solution properties and the case of nonzero input signals. The focus is on the state, $x(t)$, since the output signal, $y(t)$, easily follows from knowledge of $x(t)$ and $u(t)$.

Zero-Input Solution

For the n-dimensional linear state equation with $u(t)$, $t \ge 0$,

$$\dot{x}(t) = Ax(t), \quad x(0) = x_o$$

we assume a power series form for the solution and substitute into the equation. In this vector case, we assume a power series in t with $n \times 1$ vector coefficients, written

$$x(t) = \phi_0 + \phi_1 \frac{t}{1!} + \phi_2 \frac{t^2}{2!} + \cdots$$

At $t = 0$ this gives $\phi_0 = x_o$, and we substitute the series expression into the differential equation to obtain

$$\phi_1 + \phi_2 \frac{t}{1!} + \phi_3 \frac{t^2}{2!} + \cdots = Ax_o + A\phi_1 \frac{t}{1!} + A\phi_2 \frac{t^2}{2!} + \cdots$$

Equating coefficients of like powers of t gives

$$\begin{aligned} \phi_1 &= Ax_o \\ \phi_2 &= A\phi_1 = A^2 x_o \\ \phi_3 &= A\phi_2 = A^3 x_o \\ &\vdots \end{aligned}$$

and we can write the series expression for the solution as

$$\begin{aligned} x(t) &= x_o + Ax_o \frac{t}{1!} + A^2 x_o \frac{t^2}{2!} + \cdots \\ &= \left(I + A\frac{t}{1!} + A^2 \frac{t^2}{2!} + \cdots \right) x_o \end{aligned}$$

By analogy with the scalar case, $n = 1$, we denote the $n \times n$ matrix series in this expression by the (matrix) exponential

$$e^{At} = I + A\frac{t}{1!} + A^2 \frac{t^2}{2!} + \cdots$$

Of course this series can be viewed alternately as a power series with matrix coefficients, or as a matrix each entry of which scalar power series, though the latter view involves rather messy expressions for the scalar-entry series.

Remark
It can be shown that the series defining the matrix exponential converges uniformly in any finite interval of time, $t \in [-T, T]$, $T > 0$, and that for any finite time t the series converges absolutely.

Uniform convergence is important for justifying term-by-term calculation of the derivative and integral of the function defined by the series. Absolute convergence implies that the terms in the series can be rearranged in various ways without changing the limit function. Both of these types of manipulations on the series are used in the sequel. □

We can now reconfirm by substitution that the solution arrived at by power series substitution indeed is a solution. First we use term-by-term differentiation to compute

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt}e^{At}x_o = \frac{d}{dt}(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots)x_o \\ &= (A + A^2\frac{t}{1!} + A^3\frac{t^2}{2!} + \cdots)x_o = A(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots)x_o \\ &= Ae^{At}x_o = Ax(t)\end{aligned}$$

(The result of the term-by-term differentiation is another uniformly convergent series, and so the differentiation is valid.) Finally, evaluating at $t = 0$ shows that

$$x(0) = e^{A0}x_o = Ix_o = x_o$$

That this solution is the unique solution (for each initial state) is left to a course on differential equations.

Summing the series to compute the matrix exponential usually is not an efficient approach for computing a solution, but there are exceptions.

Example
For

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

the fact that $A^2 = 0$ gives

$$e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}t} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

This example extends in the obvious way to the computation of the exponential for any *nilpotent* matrix A.

Example
For diagonal A,

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

it is clear that every term in the series for e^{At} is diagonal, and the k^{th} diagonal term of the series is

$$1 + \lambda_k\frac{t}{1!} + \lambda_k{}^2\frac{t^2}{2!} + \cdots$$

This is the series defining the scalar exponential, and we conclude that

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Further examples of the solution of zero-input state equations will make use of properties of the matrix exponential.

Properties of the Matrix Exponential

We present several properties of e^{At} that are useful for understanding the behavior of LTI systems. All of these properties are familiar in the scalar-exponential case ($n=1$), but some care is required as not all scalar-exponential properties generalize to the matrix case. The first property in our list is essentially a reformatting of vector differential equations into a matrix differential equation that leads to a characterization of the exponential.

Property 1

For the $n \times n$ matrix differential equation, with identity initial condition,

$$\dot{X}(t) = AX(t) , \quad X(0) = I$$

the unique solution is

$$X(t) = e^{At}$$

Proof

An easy term-by-term differentiation of the series for the claimed solution shows that the matrix exponential is a solution of the matrix differential equation that satisfies the initial condition. Uniqueness of this solution, for the given matrix initial condition, follows from uniqueness of state vector equation solutions by considering each column of the matrix equation.

Property 2

Given $n \times n$ matrices A and F , a necessary and sufficient condition that

$$e^{At} e^{Ft} = e^{(A+F)t} , \quad t \geq 0$$

is that the matrices commute, $AF = FA$.

Proof

The right side of the claimed expression can be written as

$$\begin{aligned} e^{(A+F)t} &= I + (A+F)\frac{t}{1!} + (A+F)^2 \frac{t^2}{2!} + \cdots \\ &= I + (A+F)\frac{t}{1!} + (A^2 + AF + FA + F^2)\frac{t^2}{2!} + \cdots \end{aligned}$$

and the left side is

$$\begin{aligned} e^{At} e^{Ft} &= \left(I + A\frac{t}{1!} + A^2 \frac{t^2}{2!} + \cdots \right)\left(I + F\frac{t}{1!} + F^2 \frac{t^2}{2!} + \cdots \right) \\ &= I + F\frac{t}{1!} + F^2 \frac{t^2}{2!} + A\frac{t}{1!} + AF\frac{t^2}{1!} + A^2 \frac{t^2}{2!} + \cdots \\ &= I + (A+F)\frac{t}{1!} + (A^2 + 2AF + F^2)\frac{t^2}{2!} + \cdots \end{aligned}$$

If $AF = FA$, then the coefficients of like powers of t in the two expressions are identical (though most terms are not displayed). On the other hand, if the two expressions are identical, differentiating twice (term-by-term, using uniform convergence) and evaluating at $t = 0$ gives

$$A^2 + AF + FA + F^2 = A^2 + 2AF + F^2$$

from which it follows that $AF = FA$.

Property 3

For every t_1 and t_2,

$$e^{At_1} e^{At_2} = e^{A(t_1 + t_2)}$$

Proof

This follows from Property 2 by considering At_1 as "A," and At_2 as "F." □

Property 4

For any value of t, e^{At} is invertible, and

$$\left(e^{At}\right)^{-1} = e^{-At}$$

Proof

For any t we can compute the product of e^{At} and $e^{-At} = e^{A(-t)}$, in either order, by Property 3, taking $t_1 = t$ and $t_2 = -t$. This gives the $n \times n$ identity matrix as the product, and so the two matrices are inverses of each other.

Property 5

For any $n \times n$, invertible matrix P,

$$e^{P^{-1}APt} = P^{-1} e^{At} P$$

Proof

The easy fact that

$$(P^{-1}AP)^k = P^{-1}A^k P\,, \quad k = 0,1,\ldots$$

leads to

$$\begin{aligned} e^{P^{-1}APt} &= I + (P^{-1}AP)\frac{t}{1!} + (P^{-1}AP)^2 \frac{t^2}{2!} + \cdots \\ &= P^{-1}(I + A\frac{t}{1!} + A^2 \frac{t^2}{2!} + \cdots)P \\ &= P^{-1} e^{At} P \end{aligned}$$

Example

This provides a method to compute the matrix exponential when A has distinct eigenvalues. For in this situation we can compute invertible P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and then use our earlier example to conclude

$$e^{P^{-1}APt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

From this, Property 5 yields

$$e^{At} = P^{-1} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P$$

a result that displays each entry of e^{At} as a linear combination of exponentials of the eigenvalues of A.

Remark

A standard method for solving linear differential equations is to use the Laplace transform (unilateral Laplace transform since we consider signals defined for $t \geq 0$). For a matrix (or vector) function of time we define the Laplace transform to be the matrix of transforms of the entries – this preserves the linearity and derivative properties of the Laplace transform, as is easily seen by writing the following calculations out in scalar terms and repacking the results into matrix form. the Laplace transform of both sides of the $n \times n$ matrix differential equation

$$\dot{X}(t) = AX(t), \quad X_o = I$$

gives the algebraic equation

$$sX(s) - X_o = AX(s)$$

This gives $X(s) = (sI - A)^{-1} X_o$, and using the identity initial condition we conclude the following.

Property 6

The Laplace transform of the matrix exponential is given by

$$L\left\{e^{At}\right\} = (sI - A)^{-1}$$

A general, and rather detailed, representation for the matrix exponential in terms of eigenvalues of A can be obtained from the Laplace transform of the exponential and the partial fraction expansion approach to computing the inverse Laplace transform. We can write

$$(sI - A)^{-1} = \frac{\mathrm{adj}(sI - A)}{\det(sI - A)}$$

where the denominator is a degree-n polynomial in s, the characteristic polynomial of A. Furthermore, since the entries of the adjoint matrix in the numerator are computed from determinants of $(n-1) \times (n-1)$ submatrices of $(sI - A)$, in particular the cofactors of entries, it follows that each entry of the numerator is a polynomial in s of degree at most $n-1$. Thus we see that each entry of $(sI - A)^{-1}$ is a strictly-proper rational function of s, and of course the inverse transform of each entry can be computed by partial fraction expansion. Suppose

$$\det(sI - A) = (s - \lambda_1)^{m_1} \cdots (s - \lambda_l)^{m_l}$$

where $\lambda_1, \ldots, \lambda_l$ are the distinct eigenvalues of A with corresponding multiplicities $m_1, \ldots, m_l \geq 1$. Some of the distinct eigenvalues may be complex, but then the conjugates also are distinct eigenvalues, with the same multiplicity.

Now consider performing a partial fraction expansion on each of these entries. Every entry will be written as a linear combination of the terms

$$\frac{1}{(s-\lambda_k)^j}\,,\quad k=1,\ldots,l\,;\ j=1,\ldots,m_k$$

of course typically with a number of the coefficients zero. Indeed, we can gather the coefficients into matrices and write the result as a ‘matrix partial fraction expansion’

$$(sI-A)^{-1}=\sum_{k=1}^{l}\sum_{j=1}^{m_k}W_{kj}\,\frac{1}{(s-\lambda_k)^j}$$

Each W_{kj} is an $n\times n$ matrix of partial fraction expansion coefficients, and coefficients corresponding to complex λ_k ’s appearing in an entry of W_{kj} will be complex if λ_k actually appears as a demoninator root of the corresponding entry of $(sI-A)^{-1}$. One can also present formulas for the W_{kj} matrices, based on formulas for partial fraction coefficients, but we simply take the inverse Laplace transform to arrive at the expression

$$e^{At}=\sum_{k=1}^{l}\sum_{j=1}^{m_k}W_{kj}\,\frac{t^{j-1}}{(j-1)!}\,e^{\lambda_k t}$$

Again, complex conjugate terms in this expression can be combined to give a real expression for the matrix exponential, but the form given above is usually most useful.

The last property we discuss is another “finite” expression for the matrix exponential in the time domain. It is rather less specific than the eigenvalue-of- A representations, but is sometimes useful for theoretical purposes.

Property 7

There exist n scalar functions $\alpha_0(t)\,,\ldots,\,\alpha_{n-1}(t)$ such that, for all t ,

$$e^{At}=\sum_{k=0}^{n-1}\alpha_k(t)A^k$$

Proof

The Cayley-Hamilton theorem states that if

$$\det(\lambda I-A)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0$$

then

$$A^n+a_{n-1}A^{n-1}+\cdots+a_1A+a_0I=0$$

(“A square matrix satisfies its own characteristic equation.”) This means that A^n can be written as a linear combination of lower powers of A:

$$A^n=-a_0I-a_1A-\cdots-a_{n-1}A^{n-1}$$

Multiplying through by A, and replacing the A^n term on the right shows that A^{n+1} can be written as a linear combination of the same lower powers of A:

$$A^{n+1}=(a_{n-1}a_0)I-(a_0-a_{n-1}a_1)A-\cdots-(a_{n-2}-a_{n-1}^2)A^{n-1}$$

Continuing this process shows that all higher powers of A can be written as linear combinations of I , $A,\ldots,\,A^{n-1}$. Using this result in the series

$$e^{At}=I+A\frac{t}{1!}+A^2\frac{t^2}{2!}+\cdots+A^k\frac{t^k}{k!}+\cdots$$

allows the series to be written in terms of the first $n-1$ powers of A. By absolute convergence, we can gather together all the terms involving I , all the terms involving A, and so on. The coefficients of these powers of A are (admittedly messy) scalar series in t that define the functions $\alpha_0(t),\ldots,\alpha_{n-1}(t)$. □

It is worthwhile to state one *non-property*. Namely, for $n \ge 2$ it is *never* the case that the matrix exponential is formed by exponentiating each entry of A.

Solution for Nonzero Input Signal

For a linear state equation (again temporarily ignoring the output equation)

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_o$$

with an input signal $u(t), t \ge 0$, that is, for example, piecewise continuous, there are a number of approaches to deriving the solution, including integrating factor methods. However, we take a shortcut and simply guess a solution and verify.

Theorem
Given an input signal and an initial state, the complete solution to the state equation is

$$x(t) = e^{At}x_o + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\,, \quad t \ge 0$$

Furthermore, this solution is the unique solution.

Proof
Clearly this solution formula satisfies the initial condition, and the formula can be rewritten as

$$x(t) = e^{At}x_o + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau\,, \quad t \ge 0$$

to simplify differentiation. Using the product rule and a fundamental theorem of calculus,

$$\begin{aligned}\dot{x}(t) &= Ae^{At}x_o + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau + e^{At}e^{-At}Bu(t)\\ &= A\left(e^{At}x_o + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau\right) + Bu(t)\\ &= Ax(t) + Bu(t)\end{aligned}$$

The following contradiction argument shows that for a given x_o and a given input signal $u(t), t \ge 0,$ this solution is unique. Indeed, if there are two solutions, the difference between the solutions satisfies the zero-input state equation with zero initial state, and uniqueness of solutions there implies that the difference is identically zero. □

When an output equation is in play, the unique solution for the output signal is

$$y(t) = Ce^{At}x_o + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t), \quad t \ge 0$$

Of course, another way of writing this is to use the sifting property of the unit-impulse function to build the "D-term" into the integrand:

$$y(t) = Ce^{At}x_o + \int_0^t \left(Ce^{A(t-\tau)}B + D\delta(t-\tau)\right)u(\tau)\,d\tau\,, \quad t \ge 0$$

Here it is customary to call

$$h(t) = Ce^{At}B + D\delta(t)$$

the *unit-impulse response* of the state equation, though this is literally the (zero-state) response to a unit-impulse input only if one is willing to skirt the technical issue of interpreting the calculation

$$\int_0^t \delta(t-\tau)\delta(\tau)\,d\tau = \delta(t)$$

In any case, the component of the response due to the input signal can be recognized as the familiar *convolution* of the unit-impulse response and the input signal.

Using either representation, the complete solution for the output signal is a sum of a term due to the initial state, called the *zero-input term*, and a term or terms due to the input signal, called the *zero-state term(s)*.

Remark
Clearly linear state equations are blessed! No matter how large n might be, or what the coefficient matrices might be, for any given initial state and (continuous) input signal there exists a unique solution for $x(t)$ and, of course, $y(t)$. Note that this does not hold for the linear algebraic equation $Ax = b$, even for $n = 1$. □

These various response formulas also can be expressed in terms of Laplace transforms, either by solving the differential equation by transform methods, or by transforming the time-domain solution formulas. Taking the latter approach, and using the standard capital letter notation for transforms, the linearity and convolution properties of the Laplace transform yield

$$Y(s) = C(sI-A)^{-1}x_o + \left[C(sI-A)^{-1}B + D\right]U(s)$$

Again, it is customary to call

$$H(s) = C(sI-A)^{-1}B + D$$

the *transfer function* of the state equation. It is the Laplace transform of the unit-impulse response, and, from another viewpoint, the ratio of the Laplace transforms of the output signal to the input signal under the assumption of zero initial state. (Of course, for vector input and/or output signals, this should be rephrased since the transfer function will be a matrix of proper rational functions.)

Finally we confirm that the solutions for the output of two state equations related by a state-variable change are the same. That is, given an input $u(t)$ and initial state x_o for

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

the solution is the same as the solution of

$$\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$
$$y(t) = CPz(t) + Du(t)$$

to the input $u(t)$ and initial state $z(0) = P^{-1}x_o$. Indeed, using Property 5,

$$CPe^{P^{-1}APt}z(0) + \int_0^t CPe^{P^{-1}AP\tau}P^{-1}Bu(t-\tau)\,d\tau + Du(t)$$

$$= CPP^{-1}e^{At}PP^{-1}x_o + \int_0^t CPP^{-1}e^{A\tau}PP^{-1}Bu(t-\tau)\,d\tau + Du(t)$$

$$= Ce^{At}x_o + \int_0^t Ce^{A\tau}Bu(t-\tau)\,d\tau + Du(t)$$

Exercises

1. For the linear state equation with

$$A=\begin{bmatrix}-1 & 0\\ -6 & 2\end{bmatrix}\quad,\quad B=\begin{bmatrix}1/2\\ -1\end{bmatrix}\quad,\quad C=\begin{bmatrix}6 & 0\end{bmatrix}$$

use a change of state variable to compute a diagonal form state equation.

2. For linear state equation

$$\dot{x}(t)=Ax(t)+Bu(t)$$
$$y(t)=Cx(t)+Du(t)$$

consider a time-variable change of variables of the form

$$z(t)=P^{-1}(t)\,x(t)$$

What assumptions are needed on $P(t)$? What is the form of the state equation in $z(t)$?

3. Following Problem 2, investigate the case where $P(t)=e^{At}$, for the state equation $\dot{x}(t)=Ax(t)$. Comment on any special features of the resulting state equation in the new state variable.

4. Use the diagonal form method to compute e^{At} for

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

5. Use the Laplace transform method to compute e^{At} for

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

6. Show that if A is invertible, then

$$\int_0^t e^{A\sigma}\,d\sigma=A^{-1}(e^{At}-I)$$

7. Prove the transposition property

$$e^{A^Tt}=\left(e^{At}\right)^T$$

8. If A and F are $n\times n$ matrices, show that

$$e^{(A+F)t}-e^{At}=\int_0^t e^{A(t-\sigma)}Fe^{(A+F)\sigma}d\sigma$$

9. Show that under appropriate assumptions on the $n\times n$ matrix function $F(t)$,

$$\tfrac{d}{dt}e^{F(t)}=\dot{F}(t)\,e^{F(t)}$$

10. For the time-invariant, *n*-dimensional, *nonlinear* state equation

$$\dot{x}(t) = Ax(t) + Dx(t)u(t) + Bu(t)\,, \quad x(0) = 0$$

show that under appropriate additional hypotheses a solution is

$$x(t) = \int_0^t e^{A(t-\sigma)} e^{D\int_\sigma^t u(\tau)d\tau} Bu(\sigma)d\sigma$$

11. Use a power series approach to find a solution $X(t)$ for the $n \times n$ matrix differential equation

$$\dot{X}(t) = AX(t) + X(t)F\,, \quad X(0) = X_o$$

12. Using the result of Problem 11, what condition on X_o will guarantee that he $n \times n$ matrix differential equation

$$\dot{X}(t) = AX(t) - X(t)A\,, \quad X(0) = X_o$$

has a constant solution?

13. By direct differentiation of

$$x(t) = e^{At}x_o + \int_0^t e^{A(t-\sigma)}Bu(\sigma)\,d\sigma$$

show that

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_o$$

14. Is there a matrix *A* such that

$$e^{At} = \begin{bmatrix} e^t & t \\ 0 & e^{-t} \end{bmatrix}$$

15. If λ, p is an eigenvalue-eigenvector pair for A, determine a simplified form for the solution of

$$\dot{x}(t) = Ax(t)\,, \quad x(0) = p$$

If λ, and thus p, are complex, show how to interpret your result as a simplified form for the solution to a *real* initial state.

16. Show that a solution of the $n \times n$ matrix differential equation

$$\dot{X}(t) = AX(t) + X(t)F\,, \quad X(0) = X_o$$

is $X(t) = e^{At}X_o e^{Ft}$.

17. Suppose that for given *n x n* matrices *A* and *M* there exists a constant *n x n* matrix *Q* that satisfies

$$A^T Q + QA = -M$$

Show that for all $t \geq 0$,

$$Q = e^{A^T t}Qe^{At} + \int_0^t e^{A^T \sigma}Me^{A\sigma}d\sigma$$

3. Response Properties

LTI Properties of the Response

We begin by confirming linearity and time invariance properties of the response of a linear state equation. Linearity can be approached in terms of either the time domain or transform-domain solution formulas, and here the time domain is chosen:

$$y(t) = Ce^{At}x_o + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

It is obvious that the zero-input response is linear in the initial state, that is, superposition holds. If $y_a(t)$ and $y_b(t)$ are the zero-input responses to x_{oa} and x_{ob}, respectively, then for any real α the zero-input response to $x_{oa} + \alpha x_{ob}$ is $y_a(t) + \alpha y_b(t)$, $t \ge 0$. Similarly, if $y_a(t)$ and $y_b(t)$ are the zero-state responses to input signals $u_a(t)$ and $u_b(t)$, then the zero-state response to the input signal $u_a(t) + \alpha u_b(t)$ is $y_a(t) + \alpha y_b(t)$, $t \ge 0$. (These properties of course hold also for the state response, $x(t)$.

The time-invariance property of the zero-input response is a bit more subtle. If the initial state is postulated at some time $t_o > 0$, the solution of the zero-input state equation

$$\dot{x}(t) = Ax(t)\,, \quad x(t_o) = x_o$$

is

$$x(t) = e^{A(t-t_o)}x_o\,, \quad t \ge t_o$$

as is readily verified. Thus the response to shifting the initial time to t_o is a shift of the response by t_o, and this applies as well to the (zero-input) output signal,

$$y(t) = Ce^{At}x_o$$

If $y_a(t)$ is the zero-state response to the input signal $u_a(t)$, for any $t > t_o$ consider the input signal

$$u_b(t) = u_a(t-t_o)u_{step}(t-t_o)$$

where the unit-step function is used to emphasize that $u_b(t)$ is zero for $0 \le t < t_o$. (Also, we could write $u_a(t)u_{step}(t)$ to emphasize that $u_a(t) = 0$ for $t < 0$.) To conclude time invariance, we must show that the zero-state response to $u_b(t)$ is given by

$$y_b(t) = y_a(t-t_o)\,, \quad t \ge t_o$$

Indeed, for $t \ge t_o$,

$$\begin{aligned} y_b(t) &= \int_0^t Ce^{A(t-\tau)}Bu_b(\tau)\,d\tau + Du_b(t) \\ &= \int_0^t Ce^{A(t-\tau)}Bu_a(\tau-t_o)u_{step}(\tau-t_o)\,d\tau + Du_a(t-t_o)u_{step}(t-t_o) \\ &= \int_{t_o}^t Ce^{A(t-\tau)}Bu_a(\tau-t_o)\,d\tau + Du_a(t-t_o)u_{step}(t-t_o) \end{aligned}$$

Changing the variable of integration from τ to $\sigma = \tau - t_o$ gives

$$y_b(t) = \int_0^{t-t_o} Ce^{A(t-t_o-\tau)}Bu_a(\sigma)\,d\sigma + Du_a(t-t_o)u_{step}(t-t_o)$$
$$= y_a(t-t_o)$$
A completely similar argument applies to the (zero-state) state response, $x(t)$.

Response Properties Associated with Poles and Zeros

Suppose $G(s)$ is a proper rational function - a ratio of polynomials in s with the degree of the numerator polynomial no greater than the degree of the denominator polynomial. Assume also that the numerator and denominator polynomials have no roots in common, that is, they are relatively prime. A root of the numerator polynomial is called a *zero* of $G(s)$ and a root of the denominator polynomial is called a *pole* of $G(s)$. The *multiplicity* of a zero or pole s_o is the multiplicity of s_o as a root of the numerator or denominator polynomial, as appropriate.
The terms pole and zero are graphic in the sense that if s_o is a zero of $G(s)$, then $G(s_o) = 0$, and if s_o is a pole of $G(s)$, then $|G(s_o)| = \infty$. A slight subtlety has to do with the relatively-prime assumption.

Example
For
$$G(s) = \frac{s^2 - 1}{s(s+1)^2}$$
the poles are $0, -1$, both of multiplicity one, and 1 is a zero, of multiplicity one.

Remark
In the sequel we will sometimes write a transfer function with equal numerator and denominator degrees as a strictly-proper rational function plus a constant, for example,
$$G_p(s) = G_{sp}(s) + D\,, \quad D \neq 0$$
This is easily accomplished by dividing the denominator polynomial of $G_p(s)$ into the numerator polynomial for one step. To be terribly explicit, given a transfer function $G_p(s)$ with equal-degree-n numerator and denominator polynomials, written with the denominator a monic polynomial, the expression
$$\frac{c_{n-1}s^{n-1} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D = \frac{Ds^n + (c_{n-1} + a_{n-1}D)s^{n-1} + \cdots + (c_0 + a_0 D)}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$
permits the identification of $a_0, \ldots, a_{n-1}$ from the denominator of $G_p(s)$, D from the coefficient of s^n in the numerator, and then the easy calculation of $c_0, \ldots, c_{n-1}$ to define $G_{sp}(s)$. In any case, it is important to note, and easy to verify, that the poles of $G_p(s)$ are identical to the poles of $G_{sp}(s)$. However, the zeros of $G_p(s)$ and $G_{sp}(s)$ differ. □

Poles and zeros of the transfer function corresponding to a linear state equation
$$G(s) = C(sI - A)^{-1}B + D$$
can be used to characterize a number of properties of the state equation. In particular there are exponential response properties associated to each. However, another subtlety that arises is that poles of such a transfer function must be eigenvalues of A, as application of the adjoint-over-

determinant formula clearly shows, but eigenvalues of A need not be poles of the transfer function. And while zeros cannot also be poles, zeros can be eigenvalues of A.

Example
For the state equation with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a quick calculation shows that -1 is the only pole of the corresponding transfer function, but the eigenvalues of A are $-1, 1$.

Theorem
Suppose that s_o is a pole of the transfer function $G(s)$ of the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Then there exists a (possibly complex) nonzero initial state x_o and a (possibly complex) number y_o such that the zero-input output response of the state equation is

$$Ce^{At}x_o = y_o e^{s_o t}\ , \quad t \geq 0$$

Proof
Let x_o be an eigenvector corresponding to the eigenvalue s_o of A. Then

$$\begin{aligned} e^{At}x_o &= (I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots)x_o \\ &= x_o + Ax_o\frac{t}{1!} + A^2 x_o\frac{t^2}{2!} + \cdots \\ &= x_o + s_o x_o\frac{t}{1!} + s_o{}^2 x_o\frac{t^2}{2!} + \cdots \\ &= (1 + s_o\frac{t}{1!} + s_o{}^2\frac{t^2}{2!} + \cdots)x_o \\ &= e^{s_o t}x_o \end{aligned}$$

Thus

$$Ce^{At}x_o = Cx_o e^{s_o t}\ , \quad t \geq 0$$

and we can set $y_o = Cx_o$

Remark
In the case where s_o is complex, so that x_o is complex, this theorem implies a real (actual) response property as follows. Writing s_o in rectangular form as $s_o = \sigma_o + j\omega_o$, we have that

$$e^{s_o t} = e^{(\sigma_o + j\omega_o)t} = e^{\sigma_o t}\,(\cos(\omega_o t) + j\sin(\omega_o t))$$

Writing x_o in rectangular form also, we have that

$$Ce^{At}(\mathrm{Re}\{x_o\} + j\,\mathrm{Im}\{x_o\}) = C(\mathrm{Re}\{x_o\} + j\,\mathrm{Im}\{x_o\})e^{\sigma_o t}\,(\cos(\omega_o t) + j\sin(\omega_o t))\ , \quad t \geq 0$$

Expanding both sides and equating the real parts, and then the imaginary parts, gives the two real expressions for the zero-input responses to the initial states $\mathrm{Re}\{x_o\}$ and $\mathrm{Im}\{x_o\}$:

$$Ce^{At}\operatorname{Re}\{x_o\} = C\operatorname{Re}\{x_o\}e^{\sigma_o t}\cos(\omega_o t) - C\operatorname{Im}\{x_o\}e^{\sigma_o t}\sin(\omega_o t)\,, \quad t \ge 0$$

$$Ce^{At}\operatorname{Im}\{x_o\} = C\operatorname{Re}\{x_o\}e^{\sigma_o t}\sin(\omega_o t) + C\operatorname{Im}\{x_o\}e^{\sigma_o t}\cos(\omega_o t)\,, \quad t \ge 0$$

The additional complication might be viewed as sufficient reason to accept the complex formula.

In the following result concerning zeros, the assumption that the zero is not an eigenvalue of A can be removed by making use of notions that arise in the sequel.

Theorem
Suppose that s_o is a zero of the transfer function $G(s)$ of the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

but that s_o is not an eigenvalue of A. Then given a (possibly complex) number u_o there exists a (possibly complex) initial state x_o such that the complete output response of the state equation to $x(0) = x_o$ and the input signal $u(t) = u_o e^{s_o t}$ is identically zero. that is

$$Ce^{At}x_o + \int_0^t Ce^{A(t-\tau)}Bu_o e^{s_o\tau}d\tau + Du_o e^{s_o t} = 0\,, \quad t \ge 0$$

Proof
Phrasing matters in terms of Laplace transforms, we need to find x_o such that

$$Y(s) = C(sI - A)^{-1}x_o + C(sI - A)^{-1}B\,\frac{u_o}{s - s_o} + D\,\frac{u_o}{s - s_o} = 0$$

Choosing

$$x_o = (s_o I - A)^{-1}Bu_o$$

where the indicated inverse exists since s_o is not an eigenvalue of A, we can write $Y(s)$ as

$$Y(s) = C[(sI - A)^{-1}(s_o I - A)^{-1} + (sI - A)^{-1}\,\frac{1}{s - s_o}]Bu_o + D\,\frac{u_o}{s - s_o}$$

However, it is easy to verify, by multiplying on the left by $(sI - A)$, invertible for all but at most n values of the complex variable s, and on the right by $(s_o I - A)$, invertible by assumption, that

$$(sI - A)^{-1}(s_o I - A)^{-1} + (sI - A)^{-1}\,\frac{1}{s - s_o} = (s_o I - A)^{-1}\,\frac{1}{s - s_o}$$

and thus

$$\begin{aligned} Y(s) &= C(s_o I - A)^{-1}\,\frac{1}{s - s_o}Bu_o + D\,\frac{u_o}{s - s_o} \\ &= G(s_o)\frac{u_o}{s - s_o} = 0 \end{aligned}$$

Steady-State Frequency Response Properties

Given the response $y(t)$ of an LTI system to an initial state and input signal, the corresponding *steady-state response* $y_{ss}(t)$ is that function of time satisfying the following condition. Given any $\varepsilon > 0$ there exists a $T > 0$ such that

$$|\,y(t) - y_{ss}(t)\,| < \varepsilon\,, \quad \textit{for } t \ge T$$

If $y_{ss}(t)$ is a constant, then it is often called the *final value* of the response. However, $y_{ss}(t)$ need not be a constant, and also it need not be a bounded function.

Example
Consider the scalar state equation

$$\dot{x}(t) = x(t) + u(t)$$
$$y(t) = x(t)$$

with $x(0) = 0$. With the input signal

$$u(t) = 2e^{-t}$$

the Laplace transform of the response is

$$Y(s) = \frac{2}{(s+1)(s-1)} = \frac{1}{s-1} - \frac{1}{s+1}$$

and the steady-state response is

$$y_{ss}(t) = e^{t}$$

If the input signal is $u(t) = 2e^{-t} - 1, t \geq 0$, which has Laplace transform

$$U(s) = \frac{s-1}{s(s+1)}$$

then the final value of the response is

$$y_{ss}(t) = 1$$

□

Of particular importance is the following result that deals with the steady-state response to sinusoidal input signals (under an added condition on the state equation that will be developed as a stability property in the sequel).

Theorem
Suppose the LTI state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is such that all eigenvalues of A have negative real parts. Then for any initial state x_o and an input signal

$$u(t) = u_o \sin(\omega_o t) , \quad t \geq 0$$

where $\omega_o \neq 0$, the steady-state response is

$$y_{ss}(t) = u_o \,|\, G(j\omega_o) \,|\sin[\omega_o t + \angle G(j\omega_o)]$$

where

$$G(s) = C(sI - A)^{-1} B + D$$

Proof
The strategy is to compute the Laplace transform of the response, and then ignore those terms in the partial fraction expansion that correspond to components of the response that decay to zero. It is convenient to write the input signal in terms of its complex exponential components,

$$u(t) = \frac{u_o}{2j}\, e^{j\omega_o t} - \frac{u_o}{2j}\, e^{-j\omega_o t}$$

and the Laplace transform

$$U(s) = \frac{u_o}{2j}\,\frac{1}{s - j\omega_o} - \frac{u_o}{2j}\,\frac{1}{s + j\omega_o}$$

Then, given any initial state, the Laplace transform of the response is

$$Y(s) = C(sI - A)^{-1}x_o + C(sI - A)^{-1}B\frac{u_o}{2j}\,\frac{1}{s - j\omega_o} - C(sI - A)^{-1}B\frac{u_o}{2j}\,\frac{1}{s + j\omega_o}$$

$$+D\frac{u_o}{2j}\,\frac{1}{s - j\omega_o} - D\frac{u_o}{2j}\,\frac{1}{s + j\omega_o}$$

All poles of the strictly-proper rational functions $C(sI - A)^{-1}x_o$ and $C(sI - A)^{-1}B$ have negative real parts, because of the assumption on the eigenvalues of A . Therefore the partial fraction expansion terms corresponding to these poles can be ignored since they result in time functions that decay (exponentially) to zero. That is, we need only retain the partial fraction expansion terms corresponding to the poles at $\pm j\omega_o$ for the steady-state response. This gives

$$Y_{ss}(s) = C(sI - A)^{-1}B\Big|_{s=j\omega_o}\frac{u_o}{2j}\,\frac{1}{s - j\omega_o} - C(sI - A)^{-1}B\Big|_{s=-j\omega_o}\frac{u_o}{2j}\,\frac{1}{s + j\omega_o}$$

$$+D\frac{u_o}{2j}\,\frac{1}{s - j\omega_o} - D\frac{u_o}{2j}\,\frac{1}{s + j\omega_o}$$

Thus we can write

$$y_{ss}(t) = \frac{u_o}{2j}G(j\omega_o)e^{j\omega_o t} - \frac{u_o}{2j}G(-j\omega_o)e^{-j\omega_o t}$$

Expressing $G(j\omega_o)$ in polar form,

$$G(j\omega_o) = |G(j\omega_o)|e^{j\angle G(j\omega_o)}$$

and noting that since $G(s)$ is a real-coefficient, proper rational function,

$$G^*(j\omega_o) = G(-j\omega_o) = |G(j\omega_o)|e^{-j\angle G(j\omega_o)}$$

we have

$$y_{ss}(t) = u_o\,|G(j\omega_o)|\,\frac{e^{j[\omega_o t + \angle G(j\omega_o)]} - e^{-j[\omega_o t + \angle G(j\omega_o)]}}{2j}$$

$$= |G(j\omega_o)|\sin[\omega_o t + \angle G(j\omega_o)]$$

Exercises

1. The input-output behavior (zero-state output response), defined for $t \geq 0$, for various systems is given below. Determine which systems have linear input-output behavior and which have time-invariant input-output behavior.

(a) $y(t) = \int_0^t e^{t+\sigma}u(\sigma)\,d\sigma$ (b) $y(t) = 1 + \int_0^t e^{t-\sigma}u(\sigma)\,d\sigma$

(c) $y(t) = \left[\int_0^t u^3(\sigma)\,d\sigma\right]^{1/3}$

2. For the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

and positive real numbers λ_1, λ_2, suppose $y(t)$ is the response for $x(0) = 0, \quad u(t) = e^{-\lambda_1 t}$. Show that

$$\int_0^\infty y(t)e^{-\lambda_2 t}\, dt = \frac{1}{\lambda_1+\lambda_2} C(\lambda_2 I - A)^{-1} B$$

Are any additional assumptions needed? (*Hint:* A helpful notation is to define

$$h(t) = \begin{cases} Ce^{At}B, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

(the unit-impulse response) and compute $y(t)$ in terms of this.)

3. Show that two linear state equations that are related by an invertible change of state variables have the same transfer function.

4. Consider a linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_o$$
$$y(t) = Cx(t)$$

with constant input $u(t) = u_o$, $t \geq 0$. Under suitable assumptions show that the steady-state response is a constant, and derive a convenient formula for that constant.

5. For the linear state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} -3 & 1 \end{bmatrix} x(t)$$

compute the steady-state response to the input signals ($t \geq 0$)

(a) $u(t) = \delta(t)$ (b) $u(t) = 1$ (c) $u(t) = e^{-3t}$

(d) $u(t) = e^{3t}$ (e) $u(t) = 4\sin(2t)$

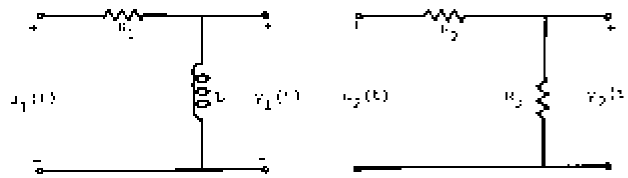
6. For the linear state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),\ x(0) = 0$$
$$y(t) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} x(t)$$

compute the steady-state response, if it exists, for the inputs ($t \geq 0$)

(a) $u(t) = \delta(t)$ (b) $u(t) = 1$ (c) $u(t) = e^{-t}$ (d) $u(t) = e^{t}$ (e) $u(t) = 4\sin(2t)$
(For (e) you need not compute the constants, simply give the form of the steady-state response.)

7. Consider the two electrical circuits with input voltages and output voltages as shown below. Compute the transfer functions $V_1(s)/U_1(s)$ and $V_2(s)/U_2(s)$. Then compute the transfer function of the cascade connection of the circuits ($u_2(t) = y_1(t)$). Is it the product of the subsystem transfer functions?



8. Suppose A is $n \times n$ and $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$.
Verify the formula

$$\operatorname{adj}(sI - A) = (s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1)I + \cdots + (s + a_{n-1})A^{n-2} + A^{n-1}$$

and use it to show that there exist strictly-proper rational functions of s such that

$$(sI - A)^{-1} = \alpha_0(s)I + \alpha_1(s)A + \cdots + \alpha_{n-1}(s)A^{n-1}$$

9. The *relative degree* of a linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is the degree of the denominator polynomial of $G(s) = C(sI - A)^{-1}B$ minus the degree of the numerator polynomial. Using Exercise 3.8, show that the state equation has relative degree κ if and only if

$$CA^{\kappa-1}B \neq 0 \;\text{ and }\; CA^kB = 0,\, k = 0,1,\ldots,\kappa-2$$

10. A transfer function $H(s)$ is such that the (zero-state) response to $\cos(2t)$ has final value zero, and the (zerp-state) response to a unit-step input is unbounded. What is a possible $H(s)$?

11. Consider the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_o$$
$$y(t) = Cx(t)$$

Is there an initial state x_o such that the response to $x(0) = x_o$, $u(t) = \delta(t)$ (the unit impulse) is $y(t) = 0$, $t \geq 0$?

12. For the linear state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x(t)$$

what are the eigenvalues of A ? What are the zeros and poles of the transfer function?

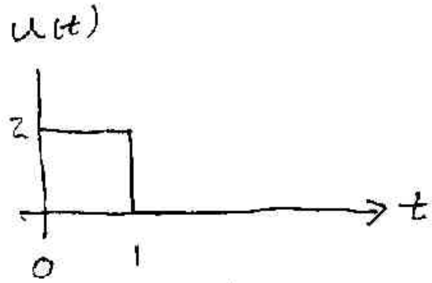
13. Suppose the zeros-state output response of the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

to a unit-step input is

$$y(t) = 3 + e^{-t} + t e^{-2t} \,, \quad t \ge 0$$

What is the zero-state output response to the input signal shown below?



14. Suppose the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is such that all eigenvalues of A have negative real parts, and suppose $u(t)$ is continuous and T-periodic for $t \ge 0$. That is, $T > 0$ is such that $u(t+T) = u(t)$ for $t \ge 0$. Show that if

$$x(0) = \int_{-\infty}^{0} e^{-A\sigma} Bu(\sigma)\, d\sigma$$

then the complete solution is T-periodic and given by

$$x(t) = \int_{-\infty}^{t} e^{A(t-\sigma)} Bu(\sigma)\, d\sigma \,, \quad t \ge 0$$

Show that the complete solution for nay other initial state converges to this periodic solution as $t \to \infty$.

4. Stability Concepts

We introduce notions that describe boundedness and asymptotic behavior of the response of a linear state equation. Since there are two components of the response, the zero-input and zero-state components, there are two categories of stability concepts. It turns out that it is useful to focus on the zero-input state response, and the zero-state output response.

Asymptotic Stability

Definition
The linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

is called *asymptotically stable* if for any initial state the zero-input state response satisfies

$$\lim_{t\to\infty} x(t) = 0$$

Theorem
The state equation (1) is asymptotically stable if and only if all eigenvalues of the coefficient matrix A have negative real parts.

Proof
For any initial state x_o, the zero-input state response is

$$x(t) = e^{At}x_o\,, \quad t \ge 0$$

If $\lambda_1, \ldots, \lambda_l$ are the distinct eigenvalues of A with corresponding multiplicities $m_1, \ldots, m_l \ge 1$, then we can write

$$e^{At} = \sum_{k=1}^{l}\sum_{j=1}^{m_k} W_{kj}\,\frac{t^{j-1}}{(j-1)!}\,e^{\lambda_k t}$$

If the eigenvalues have negative real parts, using L'Hospital's rule it is easy to show that each term in this finite summation goes to zero as $t \to \infty$, and it follows that the state equation is asymptotically stable.
Now suppose that (1) is asymptotically stable. To proceed by contradiction, suppose that there is an eigenvalue λ_1 of A with nonnegative real part. That is, $\lambda_1 = \sigma_1 + j\omega_1$, with $\sigma_1 \ge 0$. Let p be an associated eigenvector, and write p in the (vector) rectangular form

$$p = p_R + jp_I$$

Then, from Exercise 2.15, we have

$$e^{At}p = e^{\lambda_1 t}p = e^{\sigma_1 t}e^{j\omega_1 t}p$$

and this does not approach zero as $t \to \infty$ since $p \ne 0$, $e^{j\omega_1 t}$ is never zero, and $\sigma_1 \ge 0$. This completes the proof by contradiction if λ_1 is real, for then $x_o = p$ is a real initial state for which the zero-input state response does not go to zero. If λ_1 is complex, so that p is complex, writing

$$e^{At}p = e^{At}p_R + je^{At}p_I$$

makes it clear that at least one of the real initial states $x_o = p_R$ or $x_o = p_I$ yields a zero-input state response that does not go to zero. Again, this contradicts the assumption of asymptotic stability.
□

It is convenient to characterize the eigenvalue condition for asymptotic stability in terms of a linear-algebraic equation involving symmetric, sign-definite matrices called the *linear Liapunov* *equation*. The following basic result leads to connections among a number of basic properties in the sequel.

Theorem
(a) Given $n \times n$ A, if Q and M are symmetric, positive-definite matrices such that

$$QA + A^T Q = -M \tag{2}$$

then all eigenvalues of A have negative real parts.
(b) If all eigenvalues of A have negative real parts, then for each symmetric, $n \times n$ matrix M there exists a unique solution of (2) given by

$$Q = \int_0^\infty e^{A^T t} M e^{At}\, dt \tag{3}$$

Furthermore, if M is positive definite, then Q is positive definite.

Proof
(a) Suppose λ is an eigenvalue of A, with associated eigenvector p,

$$Ap = \lambda p$$

Then

$$p^H A^T = \bar{\lambda} p^H$$

where ^{H} indicates conjugate transpose. If Q and M are symmetric, positive-definite matrices such that (2) is satisfied, then

$$p^H QAp + p^H A^T Qp = -p^H Mp$$

This simplifies to

$$(\lambda + \bar{\lambda})\, p^H Qp = -p^H Mp$$

and further to

$$2\,\mathrm{Re}\{\lambda\}\, p^H Qp = -p^H Mp$$

Invoking the positive definiteness of Q and M, this implies $\mathrm{Re}\{\lambda\} < 0$.

(b) If all eigenvalues of A have negative real parts, then all entries of e^{At} and $e^{A^T t}$ go to zero exponentially as $t \to \infty$. Therefore every scalar entry in the integrand (3) similarly approaches zero, so the integral converges and Q is well defined. Further, Q is symmetric, and to show that Q is a solution of (2) we calculate

$$\begin{aligned} QA + A^T Q &= \int_0^\infty e^{A^T t} M e^{At} A\, dt + \int_0^\infty A^T e^{A^T t} M e^{At}\, dt \\ &= \int_0^\infty \frac{d}{dt}\left[e^{A^T t} M e^{At} \right] dt = e^{A^T t} M e^{At} \Big|_0^\infty \\ &= -M \end{aligned}$$

To show this solution is unique, suppose $\tilde{Q}$ is another solution. Then

$$(\tilde{Q} - Q)A + A^T(\tilde{Q} - Q) = 0$$

This gives

$$e^{A^T t}(\tilde{Q} - Q)Ae^{At} + e^{A^T t} A^T (\tilde{Q} - Q)e^{At} = 0\,, \quad t \geq 0$$

that is

$$\frac{d}{dt}\left[e^{A^T t}(\tilde{Q} - Q)e^{At} \right] = 0\,, \quad t \geq 0$$

Integrating both sides, from zero to infinity, yields

$$0 = \left[e^{A^T t} (\tilde{Q} - Q) e^{At} \right]_0^\infty = -(\tilde{Q} - Q)$$

that is, $\tilde{Q} = Q$. Finally, suppose M is positive definite. Then Q given by (3) is symmetric, and for any nonzero, $n \times 1$ x,

$$x^T Q x = \int_0^\infty x^T e^{A^T t} M e^{At} x \, dt > 0$$

since the integrand is a positive scalar function of t. Thus Q is positive definite. □

Uniform Bounded-Input, Bounded-Output Stability

For the zero-state response, the most useful concept involves boundedness of the output signal for bounded input signals. However, there is a subtlety that makes it convenient to use a concept that is a bit more complicated than 'bounded inputs yield bounded outputs.' We use the standard notion of *supremum*, where

$$\upsilon = \sup_{t \geq 0} | u(t) |$$

is the smallest constant such that $| u(t) | \leq \upsilon$ for all $t \geq 0$. If no such constant exists, we write

$$\sup_{t \geq 0} | u(t) | = \infty$$

(Notice that, for example, $u(t) = 1 - e^{-t}, t \geq 0,$ attains no maximum value, but its supremum is unity.)

Definition
The state equation is called *uniformly bounded-input, bounded-output stable* if there exists a constant η such that for any input signal $u(t)$ the zero-state output response satisfies

$$\sup_{t \geq 0} | y(t) | \leq \eta \sup_{t \geq 0} | u(t) |$$

Notice that the supremum of the zero-state output does not depend on the \`waveshape' of the input signal, only the supremum of the input, as emphasized by the adjective *uniformly*.

Theorem
The state equation is uniformly bounded-input, bounded-output stable if and only if $Ce^{At}B$ is *absolutely integrable*, that is, the integral

$$\int_0^\infty | Ce^{At} B | dt$$

is finite.

Proof
Suppose first that there is a constant ρ such that

$$\int_0^\infty | Ce^{At} B | dt = \rho$$

Then for any input signal $u(t)$, the zero-state output response satisfies

$$|y(t)| = |\int_0^t Ce^{A\tau}Bu(t-\tau)\,d\tau + Du(t)|$$

$$\leq \int_0^t |Ce^{A\tau}B||u(t-\tau)|\,d\tau + |D||u(t)|, \quad t \geq 0$$

In each term we can replace the input signal by its supremum to obtain the inequality

$$|y(t)| \leq \left(\int_0^t |Ce^{A\tau}B|\,d\tau + |D|\right)\sup_{t\geq 0}|u(t)|$$

$$\leq \left(\rho + |D|\right)\sup_{t\geq 0}|u(t)|, \quad t \geq 0$$

This implies that

$$\sup_{t\geq 0}|y(t)| \leq \left(\rho + |D|\right)\sup_{t\geq 0}|u(t)|$$

and we have shown uniform bounded-input, bounded-output stability with $\eta = \rho + |D|$.

Next, suppose the state equation is uniformly bounded-input, bounded-output stable. Then, in particular, there exists a constant η such that for any input signal satisfying

$$\sup_{t\geq 0}|u(t)| = 1$$

the corresponding zero-state output response satisfies

$$\sup_{t\geq 0}|y(t)| \leq \eta$$

We assume that the absolute-integrability condition does not hold, that is, given any constant ρ there exists a time t_ρ such that

$$\int_0^{t_\rho} |Ce^{At}B|\,dt > \rho$$

In particular this implies that there exists a time $t_\eta > 0$ such that

$$\int_0^{t_\eta} |Ce^{At}B|\,dt > \eta + |D|$$

To obtain a contradiction, consider the input signal defined by

$$u(t_\eta - \tau) = \mathrm{sgn}[Ce^{A\tau}B] = \begin{cases} 1, & Ce^{A\tau}B > 0 \\ 0, & Ce^{A\tau}B = 0 \\ -1, & Ce^{A\tau}B < 0 \end{cases}$$

for $0 \leq \tau \leq t_\eta$. Then

$$Ce^{A\tau}Bu(t_\eta - \tau) = |Ce^{A\tau}B|, \quad 0 \leq \tau \leq t_\eta$$

and the zero-state output response to the input satisfies, at $t = t_\eta$,

$$y(t_\eta) = \int_0^{t_\eta} |Ce^{A\tau}B|\,d\tau + Du(t_\eta) > \eta + |D| - |D| = \eta$$

This contradicts the assumption of uniform bounded-input, bounded-output stability. □

Input-output stability also can be phrased in terms of the transfer function of the system, the Laplace transform of the unit-impulse response. □

Theorem

The state equation is uniformly bounded-input, bounded-output stable if and only if all poles of the transfer function,

$$G(s) = C(sI - A)^{-1}B + D$$

have negative real parts.

Proof

We can assume $D = 0$ since the value of D does not impact either the stability concept or the pole locations of the transfer function. If the stability property holds, then $Ce^{At}B$ is absolutely integrable. Using a representation for the matrix exponential, we can write

$$Ce^{At}B = \sum_{k=1}^{l}\sum_{j=1}^{m_k} CW_{kj}B\frac{t^{j-1}}{(j-1)!}e^{\lambda_k t}$$

and conclude from absolute integrability that, for each j, either λ_j has negative real part, or

$$CW_{kj}B = 0\,, \quad j = 1,\ldots,m_k$$

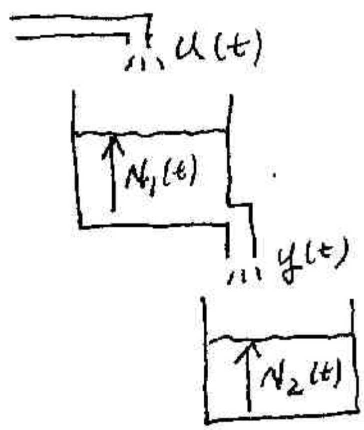
But then

$$C(sI - A)^{-1}B = \sum_{k=1}^{l}\sum_{j=1}^{m_k} CW_{kj}B\frac{1}{(s-\lambda_k)}$$

placed over a common denominator, has negative-real-part poles.

On the other hand, if $C(sI - A)^{-1}B$ has negative-real-part poles, then partial fraction expansion shows that $Ce^{At}B$ is a linear combination of (t-multiplied) *decaying* exponentials, and therefore is absolutely integrable. □

Example



The bucket system shown, with our standard assumptions and all parameters unity, and no outlet from the second bucket, is described by the linear state equation

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Since $\det[\lambda I - A] = \lambda(\lambda + 1)$, all eigenvalues of A do not have negative real parts, and the state equation is not asymptotically stable. Of course this matches our cartoon intuition, for if $x_2(0) \neq 0$, then $x_2(t)$ will never approach zero. However,

$$Ce^{At}B = e^{-t}$$

and it is straightforward to check that the state equation is uniformly bounded-input, bounded-output stable. □

Often the behavior illustrated by the example is called 'hidden instability.' This terminology is motivated by the fact that from an input-output viewpoint the system is well behaved, but internal (state) variables can grow without bound. If our example seems trivially transparent, less obvious examples of the non-equivalence of the two notions of stability abound (see Exercise 4.4). We will return to this issue in the sequel.

Exercises

1. A linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is called *stable* (or, sometimes, *marginally stable*) if the zero-input response $x(t)$ to any initial state is a bounded function of time. That is, given $x(0)$ there is a finite k such that $\|x(t)\| \le k\,,\quad t \ge 0$. What is a simple sufficient condition on A for the state equation to be stable?

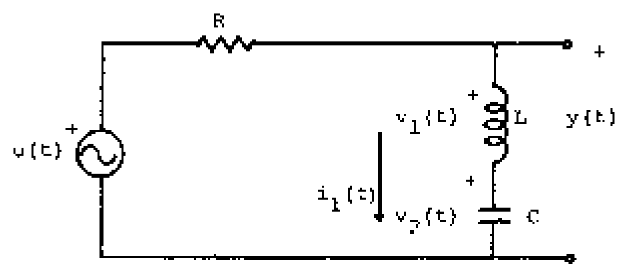
2. Apply your sufficient condition for marginal stability from Exercise 4.1 to the examples

(a) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *(b)* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

By computing the zero-input state responses, determine if the examples actually are marginally stable.

3. Show by example that a (marginally) stable linear state equation, as defined in Exercise 4.1 need not be uniformly bounded-input, bounded-output stable.

4. Consider the circuit shown below



where the circuit parameters satisfy $L, C > 0,\ \ R \ge 0$. What further conditions on the parameters are needed for asymptotic stability? For uniform bounded-input, bounded-output stability?

5. For what values of the parameter α is the linear state equation below asymptotically stable? Uniformly bounded-input, bounded-output stable?

$$\dot{x}(t) = \begin{bmatrix} 0 & \alpha \\ 2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

6. Consider the state equation $\dot{x}(t) = AF\,x(t)$, where F is a symmetric, positive-definite, $n \times n$ matrix and A is an $n \times n$ matrix such that $A + A^T$ is negative definite. Show that the state equation is asymptotically stable. (*Hint*: Begin by considering an eigenvalue/eigenvector pair for AF, form $p^H FAFp = \lambda p^H FP$, and proceed.)

7. Suppose $\dot{x}(t) = A\,x(t)$ is asymptotically stable. For what range of the scalar parameter α is

$$\dot{z}(t) = \alpha A z(t)$$

asymptotically stable?

8. For each system described below, determine if the system is uniformly bounded-input, bounded-output stable, and if not provide a bounded input that yields an unbounded output.

(a) $G(s) = \dfrac{s-1}{s^3 + 2s^2 + s}$ *(b)* $G(s) = \dfrac{s}{(s+3)(s^2+9)}$ *(c)* $G(s) = \dfrac{s}{(s+3)(s^2-9)}$

(d)
$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

9. Suppose an LTI system is described by an *improper*, rational transfer function with numerator degree $n+1$ and denominator degree n. Show that such a system cannot be uniformly bounded-input, bounded-output stable.

5. Controllability and Observability

The state equation representation exhibits the internal structure of a system, in particular the connections among the state variables and the input and output signals. If the input signal cannot influence some of the state variables, or the output signal is not influenced by some of the state variables, then we might well expect that the system has some unusual features. A particular example from Section 4 is the occurrence of uniformly bounded-input, bounded-output systems that are not asymptotically stable. This section introduces concepts that capture the relevant structural issues.

Roughly speaking, a state equation is controllable if the input signal can independently influence each of the state variables, and a state equation is observable if the state variables independently influence the output signal. These concepts, when made precise, turn out to be fundamental.

Controllability

Definition
The linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

is called *controllable* if given any initial state x_o there exists a finite time t_f and a continuous input signal $u(t)$ such that the zero-state state response satisfies $x(t_f) = 0$.

The controllability property certainly involves the dynamical behavior of the state equation, but the property can be characterized in purely algebraic terms. Indeed, we will develop three different algebraic criteria for controllability.

Theorem
The linear state equation (1) is controllable if and only if the *controllability matrix*

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{2}$$

has rank n.

Proof
We first show that if the rank condition fails, then there exist initial states that cannot be driven to the origin in finite time. If the rank condition fails, the column vectors $B, AB, \dots, A^{n-1}B$ are linearly dependent, and there exists an x_o that *cannot* be written as a linear combination of these vectors. For the purpose of obtaining a contradiction, suppose that this x_o can be driven to the origin in finite time. That is, there exists a $t_f > 0$ and a continuous input signal $u(t)$ such that

$$0 = x(t_f) = e^{At_f} x_o + \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau)\, d\tau$$

Rearranging this expression and using a finite representation for the matrix exponential (involving scalar functions $\alpha_0(t), \dots, \alpha_{n-1}(t)$)gives

$$x_o = -\int_0^{t_f} e^{-A\tau} Bu(\tau)\, d\tau$$

$$= \sum_{k-0}^{n-1} \int_0^{t_f} (-1)^{k+1} \alpha_k(\tau) u(\tau)\, d\tau\; A^k B$$

But this shows that x_o can be written as a linear combination of $B, AB, \dots, A^{n-1}B$, with coefficients defined by the scalar integrals, which is a contradiction. Thus x_o cannot be driven to the origin in finite time, and the state equation is not controllable.

Now suppose that the rank condition holds. First we show that this implies that the $n \times n$ matrix

$$\int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau \tag{3}$$

is invertible for any $t_f > 0$. Again the proof is by contradiction, and we first assume that there is a particular $t_f > 0$ such that the matrix is not invertible. This implies that there exists a nonzero, $n \times 1$ vector x such that

$$0 = x^T \int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau\; x = \int_0^{t_f} x^T e^{-A\tau} BB^T e^{-A^T\tau} x\, d\tau$$

Since the integrand is the product of the two identical scalar functions $x^T e^{-A\tau} B$ and $B^T e^{-A^T\tau} x$, we have that

$$x^T e^{-A\tau} B = 0\,, \quad 0 \le \tau \le t_f$$

Therefore,

$$x^T e^{-A\tau} B\Big|_{\tau=0} = x^T B = 0$$

$$\frac{d}{d\tau} x^T e^{-A\tau} B\Big|_{\tau=0} = -x^T A e^{-A\tau} B\Big|_{\tau=0} = -x^T AB = 0$$

$$\vdots$$

$$\frac{d^{n-1}}{d\tau^{n-1}} x^T e^{-A\tau} B\Big|_{\tau=0} = (-1)^{n-1} x^T A^{n-1} e^{-A\tau} B\Big|_{\tau=0} = (-1)^{n-1} x^T A^{n-1} B = 0$$

which gives

$$x^T A^k B = 0\,, \quad k = 0, 1, \dots, n-1$$

Rearranging this data to write

$$\begin{bmatrix} x^T B & x^T AB & \cdots & x^T A^{n-1} B \end{bmatrix} = x^T \begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix} = 0$$

makes the contradiction of the original full-rank assumption clear.

Using the invertibility of (3), given x_o we choose *any* $t_f > 0$ and the input signal

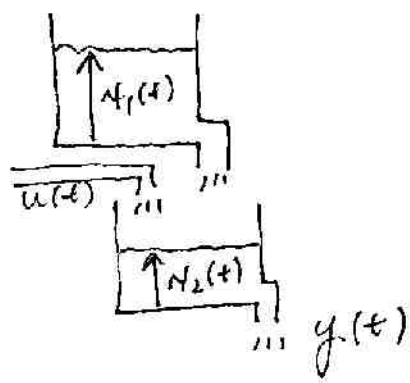
$$u(t) = -B^T e^{-A^T t} \left[\int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau \right]^{-1} x_o$$

Then the complete state response at time t_f can be computed as follows:

$$
\begin{aligned}
x(t_f) &= e^{At_f} x_o + \int_0^{t_f} e^{A(t_f-\sigma)} Bu(\sigma)\, d\sigma \\
&= e^{At_f} x_o - e^{At_f} \int_0^{t_f} e^{-A\sigma} BB^T e^{-A^T\sigma} \left[\int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau \right]^{-1} d\sigma\, x_o \\
&= e^{At_f} x_o - e^{At_f} \int_0^{t_f} e^{-A\sigma} BB^T e^{-A^T\sigma} d\sigma \left[\int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau \right]^{-1} x_o \\
&= 0
\end{aligned}
$$

Thus this input signal is as effective at demonstrating controllability as it is unmotivated! Furthermore, the transfer to the origin can be accomplished in any desired time $t_f > 0$. □

Example
The bucket system shown below, with all parameters unity,



is described by the state equation

$$
\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
$$
$$
y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)
$$

The clear intuition is that this system is not controllable, since the input cannot influence the first state variable, and indeed

$$
rank\begin{bmatrix} B & AB \end{bmatrix} = rank\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = 1
$$

Of course our cartoon intuition is dangerous in that negative values for the input signal and the state variables are apparently impossible, and the notion of controllability is not applicable to restricted classes of input signals. One reasonable fix for this is to reformulate the example in terms of deviation variables about a constant, positive input and corresponding constant, positive values of the state variables and output signal, as illustrated in Section 1. Then negative values for the deviation variables are sensible, at least within some fixed range. However, since the coefficient matrices in the linear state equation do not change in the reformulation, we ignore this issue for our bucket cartoons.

Example
Consider the diagonal state equation

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x(t) + Du(t)$$

Intuition demands that every entry of B must be nonzero for controllability, but it is far from clear whether this condition is sufficient. Turning to the rank calculation,

$$rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = rank\begin{bmatrix} b_1 & \lambda_1 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \vdots & \vdots \\ b_n & \lambda_n b_n & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

$$= rank\left(\begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \right)$$

The second of the two $n \times n$ matrices in the product is a *Vandermonde* matrix, and is known to be invertible if and only if $\lambda_1, \ldots, \lambda_n$ are distinct. Since the first matrix is diagonal, it follows that, under the distinct-eigenvalue assumption, a necessary and sufficient condition for controllability is that every entry of B be nonzero. Notice that this indicates that controllability indeed is a structural property in the sense that the property holds regardless of what the (nonzero) values of the b_k's might be, or what the (distinct) values of the λ_k's might be. □

Observability

Definition
The linear state equation (1) is called *observable* if there exists a finite time t_f such that the initial state x_o is uniquely determined by the zero-input output response $y(t)$ for $0 \le t \le t_f$.

The property of observability also can be characterized in purely algebraic terms, and again different algebraic criteria for observability are useful.

Theorem
The linear state equation (1) is observable if and only if the *observability matrix*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4}$$

has rank n.

Proof
If the rank condition does not hold, we will show that the state equation is not observable by exhibiting a nonzero initial state x_o that yields the same zero-input output response as the zero

initial state. Since the rank condition does not hold, there exists an $n \times 1$, nonzero vector x_o such that

$$0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_o = \begin{bmatrix} Cx_o \\ CAx_o \\ \vdots \\ CA^{n-1}x_o \end{bmatrix}$$

Then, using a finite representation for the matrix exponential, the zero-input response to this initial state is

$$y(t) = Ce^{At}x_o = \sum_{k=0}^{n-1} \alpha_k(t) CA^k x_o = 0\,, \quad t \geq 0$$

Thus, from the zero-input output response, x_o cannot be distinguished from the zero initial state.

Now suppose that the rank condition holds. Just as in the proof of the controllability rank condition, we can show that this implies that the $n \times n$ matrix

$$\int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

is invertible for any $t_f > 0$. Leaving this as a modest exercise, for any initial state x_o the zero-input output response is given by

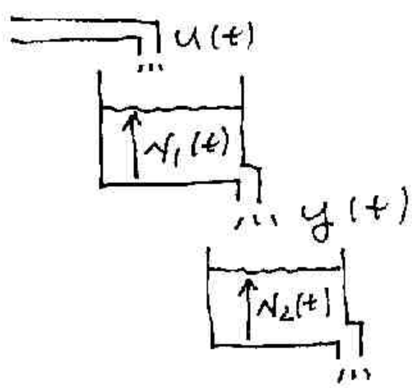
$$y(t) = Ce^{At}x_o\,, \quad t \geq 0$$

Multiplying both sides by $e^{A^T t}C^T$ and integrating from 0 to any $t_f > 0$ gives the linear algebraic equation

$$\int_0^{t_f} e^{A^T\tau} C^T y(\tau)\, d\tau = \int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau}\, d\tau\, x_o$$

Since the $n \times 1$ left side is known, and the $n \times n$ matrix on the right side is invertible, this shows that x_o is uniquely determined, regardless of the particular choice of t_f. □

Example
The bucket system shown below, with all parameters unity,



is described by the state equation

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

As expected, this system is not observable, since

$$rank \begin{bmatrix} C \\ CA \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = 1$$

Additional Controllability and Observability Criteria

Because of operational implications (steering the state with the input, or ascertaining the state from the output) and structural implications (connectedness of the input and output to the states) the concepts of controllability and observability are central to much of the material in the sequel. Alternate forms of the criteria for controllability and observability prove highly effective. Particular changes of state variables are required for the proofs, and underlying the arguments are the rather obvious facts that for two linear state equations related by a change of variables,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

and

$$\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$
$$y(t) = CPx(t) + Du(t)$$

the respective controllability and observability matrices satisfy

$$\begin{aligned} rank \begin{bmatrix} P^{-1}B & (P^{-1}AP)P^{-1}B & \cdots & (P^{-1}AP)^{n-1}P^{-1}B \end{bmatrix} &= rank \left(P^{-1} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \right) \\ &= rank \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \end{aligned}$$

and

$$rank \begin{bmatrix} CP \\ CP(P^{-1}AP) \\ \vdots \\ CP(P^{-1}AP)^{n-1} \end{bmatrix} = rank \left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} P \right) = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

In other words, controllability and observability properties are preserved under a change of state variables.

Lemma

Suppose the linear state equation (1) is such that

$$rank \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = q$$

where $0 < q < n$. (The cases $q = 0$ and $q = n$ are trivial.) Then there exists an invertible, $n \times n$ matrix P such that

$$F = P^{-1}AP = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \quad G = P^{-1}B = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{5}$$

where F_{11} is $q \times q$, G_1 is $q \times 1$, and

$$rank \begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{q-1}G_1 \end{bmatrix} = q$$

Proof

The claimed $n \times n$ matrix P can be constructed as follows. Select q linearly independent vectors, $p_1, \dots, p_q$ from the set $B, AB, \dots, A^{n-1}B$. Note here that an application of the Cayley-Hamilton theorem shows that any vector of the form $A^k B$, regardless of the nonnegative integer k, can be written as a linear combination of $p_1, \dots, p_q$. Put another way,

$$rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = rank\begin{bmatrix} B & \cdots & A^{n-1}B & \cdots & A^{n+k}B \end{bmatrix}$$

for any nonnegative integer k. Next, select any $n-q$ additional vectors $p_{q+1}, \dots, p_n$ such that

$$P = \begin{bmatrix} p_1 & \cdots & p_q & p_{q+1} & \cdots & p_n \end{bmatrix}$$

is invertible (n linearly independent columns). Now the corresponding structure of G, in particular the bottom $n-q$ zero entries, can be ascertained by inspection of the relation $PG = B$ as follows. Clearly B is given by a linear combination of columns of P, with entries of G as the coefficients. Since B can be written as a linear combination of the first q columns of P, the bottom $n-q$ entries of G are zero.

A similar argument confirms the lower-left zero partition of F, based on the relation

$$PF = AP = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix}$$

Since $A^k B$, for any $k \ge 0$, can be written as a linear combination of $p_1, \dots, p_q$, the vectors $Ap_1, \dots, Ap_q$ can be written as a linear combination of $p_1, \dots, p_q$. Thus the first q columns of F must have zeros as the bottom $n-q$ entries.

With the claimed structure of F and G confirmed, the completion of the proof is based on the partitioned product calculation

$$\begin{aligned} P^{-1}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} &= \begin{bmatrix} P^{-1}B & P^{-1}AB & \cdots & P^{-1}A^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix} \\ &= \begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{n-1}G_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

The rank of the product matrix remains q, and again from the note above we have

$$rank\begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{n-1}G_1 \end{bmatrix} = rank\begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{q-1}G_1 \end{bmatrix} = q$$

□

The change of state variables in this result can be interpreted by writing the new state variable in the partitioned form

$$\begin{bmatrix} z_c(t) \\ z_{nc}(t) \end{bmatrix} = P^{-1}x(t)$$

where $z_c(t)$ is $q \times 1$. Then the new state equation decomposes to

$$\begin{aligned} \dot{z}_c(t) &= F_{11}z_c(t) + F_{12}z_{nc}(t) + G_1u(t) \\ \dot{z}_{nc}(t) &= F_{22}z_{nc}(t) \end{aligned}$$

Clearly the second component, $z_{nc}(t)$ is not influenced by the input signal. However, it can be shown that the first subsystem is controllable, regardless of the extra term on the right side. (Exercise 5.3)

Our main use of the lemma is in proving the following characterization of controllability.

Theorem
The linear state equation (1) is controllable if and only if for every complex scalar λ the only complex $n \times 1$ vector p that satisfies

$$p^T A = \lambda p^T \ , \quad p^T B = 0 \tag{6}$$

is $p = 0$.

Proof
We will establish equivalence of the negations, that is, (6) is satisfied for some $p \neq 0$ if and only if the state equation is *not* controllable. First, if $p \neq 0$ is such that (6) is satisfied for some λ, then

$$\begin{aligned} p^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} &= \begin{bmatrix} p^T B & p^T AB & \cdots & p^T A^{n-1} B \end{bmatrix} \\ &= \begin{bmatrix} p^T B & \lambda p^T B & \cdots & \lambda^{n-1} p^T B \end{bmatrix} \\ &= 0 \end{aligned}$$

This shows that the $n \times n$ controllability matrix is not full rank, and thus the state equation is not controllable.

Next, suppose the linear state equation is not controllable. Then by the lemma there exists an invertible P such that (5) holds, and $0 < q < n$. (Again, the case $q = 0$ is trivial.) Suppose p_q is a left eigenvector for F_{22} corresponding to the eigenvalue λ. That is,

$$p_q^T F_{22} = \lambda p_q^T \ , \quad p_q^T \neq 0$$

Then with

$$p^T = \begin{bmatrix} 0_{1\times q} & p_q^T \end{bmatrix} P^{-1}$$

we have that $p \neq 0$ and

$$\begin{aligned} p^T B &= \begin{bmatrix} 0_{1\times q} & p_q^T \end{bmatrix} P^{-1} B = \begin{bmatrix} 0_{1\times q} & p_q^T \end{bmatrix} \begin{bmatrix} G_1 \\ 0 \end{bmatrix} = 0 \\ p^T A &= \begin{bmatrix} 0_{1\times q} & p_q^T \end{bmatrix} P^{-1} A = \begin{bmatrix} 0_{1\times q} & p_q^T \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 0_{1\times q} & \lambda p_q^T \end{bmatrix} P^{-1} = \lambda p^T \end{aligned}$$

and this completes the proof. □

A quick paraphrase of this result is that *a state equation is controllable if and only if there is no left eigenvector of A that is orthogonal to B*. A reformatting of the condition yields another useful form for the controllability criterion:

Theorem
The linear state equation (1) is controllable if and only if

$$rank \begin{bmatrix} sI - A & B \end{bmatrix} = n \tag{7}$$

for every complex scalar s.

Proof

Again we use the strategy of the previous proof and show that the rank condition fails if and only if the state equation is not controllable. But the state equation is not controllable if and only if there exist a scalar λ and $n \times 1$ vector $p \neq 0$ such that (6) holds. This can be rewritten as

$$p^T \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0$$

Since $p \neq 0$, this is equivalent to

$$rank \begin{bmatrix} \lambda I - A & B \end{bmatrix} < n$$

which of course is failure of the rank condition (7). □

To obtain alternate forms of the observability rank condition, there is a shortcut that bypasses the need to present proofs parallel to those above. Simply compare controllability and observability matrices, after obvious transpositions, to confirm that the linear state equation (1) is observable (respectively, controllable) if and only if the linear state equation

$$\begin{aligned} \dot{z}(t) &= A^T z(t) + C^T u(t) \\ y(t) &= B^T z(t) + Du(t) \end{aligned} \tag{8}$$

is controllable (respectively, observable). Thus the controllability results for (8) can be restated, after further transposition of coefficient matrices, as observability theorems for (1).

Lemma
Suppose the linear state equation (1) is such that

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = l$$

where $0 < l < n$. Then there exists an invertible, $n \times n$ matrix Q such that

$$F = Q^{-1} A Q = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}, \quad H = CQ = \begin{bmatrix} H_1 & 0 \end{bmatrix}$$

where F_{11} is $l \times l$, H_1 is $l \times 1$, and

$$rank \begin{bmatrix} H_1 \\ H_1 F_{11} \\ \vdots \\ H_1 F_{11}^{l-1} \end{bmatrix} = l$$

Theorem
The linear state equation (1) is observable if and only if for every complex scalar λ the only complex $n \times 1$ vector p that satisfies

$$Ap = \lambda p, \quad Cp = 0$$

is $p = 0$.

In words, (1) is observable if and only if there is no (right) eigenvector for A that is orthogonal to C.

Theorem

The linear state equation (1) is observable if and only if

$$rank\begin{bmatrix} C \\ sI-A \end{bmatrix} = n$$

for every complex scalar s .

Controllability and Observability Forms

There are special state-variable changes that are associated with the properties of controllability and observability, and these are useful for providing explicit and elementary proofs of various results in the sequel.

Definition
A linear state equation of the form

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & & \cdots & & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{9}$$

$$y(t) = \begin{bmatrix} c_0 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix} x(t) + Du(t)$$

is said to be in *controllability form*. (The " A " matrix has zero entries except for 1 's above the diagonal and possibly nonzero entries in the bottom row.)

A controllability form state equation is controllable, regardless of the values of the a_k and c_k coefficients, for the controllability matrix has the form

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -a_{n-1} \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & \sim \\ 1 & -a_{n-1} & \cdots & \sim \end{bmatrix}$$

That is, the controllability matrix is lower triangular with one's on the anti-diagonal (*don't care* entries are denoted by " $\sim$ "). Furthermore, if a linear state equation (1) is controllable, then we will show how to construct an invertible state-variable change P such that

$$\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$
$$y(t) = CPz(t) + Du(t)$$

is in controllability form. To explicitly compute P , suppose

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{10}$$

and define a set of $n \times 1$ vectors according to

$$p_0 = B$$
$$p_k = Ap_{k-1} + a_{n-k}p_0 \,, \quad k = 1, \ldots, n$$

Since $B, AB, \ldots, A^{n-1}B$ are linearly independent and p_k is a linear combination of $A^kB, A^{k-1}B, \ldots, B$, with nonzero scalar coefficient for A^kB , it follows that $p_0, p_1, \ldots, p_{n-1}$ are linearly independent. Furthermore p_n , introduced purely for notational convenience, is

$$\begin{aligned}
p_n &= Ap_{n-1} + a_0 p_0 \\
&= A^2 p_{n-2} + a_1 Ap_0 + a_0 p_0 \\
&= A^3 p_{n-3} + a_2 A^2 p_0 + a_1 Ap_0 + a_0 p_0 \\
&\vdots \\
&= A^n p_0 + a_{n-1} A^{n-1} p_0 + \cdots + a_1 Ap_0 + a_0 p_0 \\
&= 0
\end{aligned}$$

by the Cayley-Hamilton theorem.

Now let

$$P = \begin{bmatrix} p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix}$$

This $n \times n$ matrix is invertible, since the columns are linearly independent, and a partitioned multiplication by P verifies that

$$P^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

To verify that $P^{-1}AP$ has the claimed form, another partitioned multiplication gives

$$\begin{aligned}
P \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} &= \begin{bmatrix} -a_0 p_0 & p_{n-1} - a_1 p_0 & \cdots & p_1 - a_{n-1} p_0 \end{bmatrix} \\
&= \begin{bmatrix} Ap_{n-1} & Ap_{n-2} & \cdots & Ap_0 \end{bmatrix} \\
&= AP
\end{aligned}$$

Thus we have verified that the state-variable change yields a state equation in controllability form, and the coefficients on the bottom row of $P^{-1}AP$ are precisely the coefficients of the characteristic polynomial of A. (The entries of CP , the c_k 's in (9), have no particular interpretation at this point.)

Using the connection between controllability and observability properties for (1) and (8), the parallel results for observability are straightforward to work out. For the record we state the conclusions as follows.

Definition

A linear state equation of the form

$$\begin{aligned}
\dot{z}(t) &= \begin{bmatrix} 0 & & & -a_0 \\ 1 & & & \\ & \ddots & & \vdots \\ & & 0 & \\ & & 1 & -a_{n-1} \end{bmatrix} z(t) + \begin{bmatrix} b_0 \\ \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} z(t) + Du(t)
\end{aligned} \tag{11}$$

is said to be in *observability form*. (The “ A” matrix has zero entries except for 1 ’s below the diagonal and possibly nonzero entries in the right-most column.)

Such a state equation is observable, regardless of the coefficient values for the a_k ’s and b_k ’s. Furthermore, if (1) is an observable linear state equation, then there is a state-variable change that transforms it into observability form. Specifically, define a set of $1\times n$ vectors by

$$q_0 = C$$
$$q_k = q_{k-1}A + a_{n-k}q_0\ , \quad k = 1,\ldots,n$$

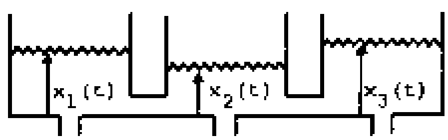
Then with

$$Q^{-1} = \begin{bmatrix} q_{n-1} \\ q_{n-2} \\ \vdots \\ q_0 \end{bmatrix}$$

the variable change $z(t) = Q^{-1}x(t)$ renders (1) into (11).

Exercises

1. Consider the linear bucket system shown below, with all parameter values unity.
(a) If the input is applied to the left tank, is the state equation controllable?
(b) If the input is applied to the center tank, is the state equation controllable?
Can you intuitively justify your conclusions?



2. For what values of the parameter *a* is the linear state equation

$$\dot{x}(t) = \begin{bmatrix} 1 & a & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} x(t)$$

controllable? Observable?

3. Consider the n-dimensional linear state equation

$$\dot{z}_c(t) = F_{11}z_c(t) + F_{12}z_{nc}(t) + G_1u(t)$$
$$\dot{z}_{nc}(t) = F_{22}z_{nc}(t)$$

where $z_c(t)$ is $q\times 1$. Show that if

$$rank\begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{q-1}G_1 \end{bmatrix} = q$$

then given any initial state $z_c(0), z_{nc}(0)$ there is an input signal $u(t)$ and a finite time $t_f > 0$ such that $z_c(t_f) = 0$.

4. Show that if the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable, then *any* initial state x_o can be transferred to *any* desired state x_d in finite time.

5. Show that if the linear, dimension-n, state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is controllable and

$$rank \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n+1$$

then the linear state equation

$$\dot{z}(t) = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} z(t) + \begin{bmatrix} B \\ D \end{bmatrix} v(t)$$

is controllable.

6. Show that the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if and only if the only $n \times n$ matrix *X* that satisfies

$$XA = AX \ , \quad XB = 0$$

is $X = 0$. (*Hint*: Employ right and left eigenvectors of *A*.)

7. Consider the *n*-dimensional linear state equation

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u(t)$$

where A_{11} is $q \times q$ and B_{11} is $q \times m$ with rank *q*. Prove that this state equation is controllable if and only if the $(n-q)$-dimensional linear state equation

$$\dot{z}(t) = A_{22}z(t) + A_{21}v(t)$$

is controllable.

8. Suppose that the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable and *A* has negative-real-part eigenvalues. Show that there exists a symmetric, positive-definite matrix *Q* such that

$$AQ + QA^T = -BB^T$$

9. Suppose that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is controllable and there exists a symmetric, positive-definite *Q* such that

$$AQ + QA^T = -BB^T$$

Show that all eigenvalues of *A* have negative real parts.

10. Give proofs or counterexamples to the following statements about the linear state equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
and
$$\dot{x}(t) = (A - BK)x(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
(a) If the first is controllable, then the second is controllable for all $1\times n$ vectors *K*.

(b) If the first is observable, then the second is observable for all $1\times n$ vectors *K*.

11. The linear state equation
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
is called *output controllable* if for any initial state there is a finite time $t_f > 0$ and a continuous input signal such that the corresponding output response satisfies $y(t_f) = 0$. Derive a necessary and sufficient condition for output controllability. Can you interpret your condition in terms of the transfer function of the state equation?

12. Show that the linear state equation of dimension $n = 2$,
$$\dot{x}(t) = Ax(t) + Bu(t)$$
is controllable for every nonzero vector *B* if and only if the eigenvalues of *A* are complex.

13. Suppose the linear state equation
$$\dot{x}(t) = Ax(t)\,, \quad x(0) = x_o$$
$$y(t) = Cx(t)$$
is observable. Show that if x_o is such that $\lim_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.

6. Realization

Given a linear state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

it is straightforward to compute the corresponding transfer function

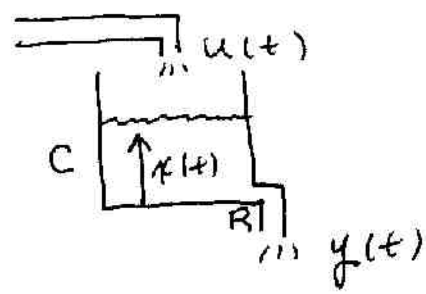
$$H(s) = C(sI - A)^{-1}B + D \tag{2}$$

or unit-impulse response

$$h(t) = Ce^{At}B + D\delta(t)$$

both of which represent the input-output behavior (zero-state output response) of the system. In this section we address the reverse: given a transfer function or unit-impulse response, characterize and compute the corresponding linear state equations. It should be clear that this is a more complex issue since we are attempting to infer the description and properties of internal variables from input-output information. There are obvious limits to what can be inferred since a change of state variables does not change the transfer function or unit-impulse response. If there is one state equation that has the given transfer function, there are an infinite number, all of the same dimension. Perhaps less obvious is the fact that there also are infinite numbers of state equations with different dimensions that have the given transfer function.

Example
Consider two bucket systems, both with all parameter values unity. The first is the single bucket



described by the scalar linear state equation

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t) \\ y(t) &= x(t)\end{aligned}$$

An easy calculation gives the corresponding transfer function

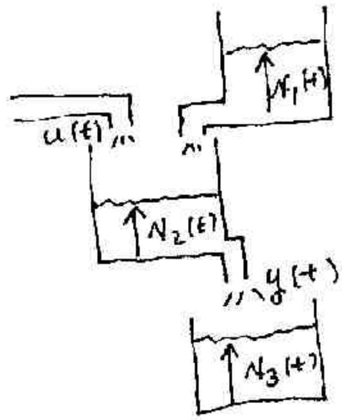
$$H(s) = \frac{1}{s+1}$$

The three-bucket system shown below is described by

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

The transfer function corresponding to this state equation is



$$H(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ -1 & s+1 & 0 \\ 0 & -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s+1}$$

This confirms our cartoon intuition that the two bucket systems have the same input-output behavior, that is, zero-state output response. □

The bucket example motivates a concentration on least-dimension state equations corresponding to a given transfer function, on the grounds of economy. To make the discussion more precise, we adopt some formal terminology. It is presented in terms of transfer function descriptions of input-output behavior, but extends naturally to unit-impulse response descriptions.

Definition
Given a transfer function $H(s)$, a linear state equation (1) is called a *realization* of $H(s)$ if

$$C(sI - A)^{-1}B + D = H(s)$$

A transfer function is called *realizable* if there exists such a realization. If (1) is a realization of $H(s)$ with dimension n and no lower-dimension realization exists for $H(s)$, then (1) is called a *minimal realization* of $H(s)$.

Realizability

An obvious first step is to characterize realizability. For this purpose, recall the definitions of proper and strictly-proper rational functions in Section 3.

Theorem
A transfer function is realizable if and only if it is a proper rational function.

Proof
If a transfer function $H(s)$ is realizable, then we can assume that (1) is a corresponding realization. From (2) and the fact from Section 2 that $(sI - A)^{-1}$ is a matrix of strictly-proper rational functions it follows easily that $H(s)$ is a proper rational function (strictly proper if $D = 0$).

Now suppose $H(s)$ is a proper rational function. From Section 3 we can write it in the form

$$\frac{Ds^n + (c_{n-1} + a_{n-1}D)s^{n-1} + \cdots + (c_0 + a_0 D)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + D \tag{3}$$

This identifies the value of D, and we next show that the (controllability form) linear state equation specified (in our shorthand notation from Section 5) by

$$A = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & & \cdots & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix} \tag{4}$$

is such that

$$C(sI - A)^{-1}B = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{5}$$

To simplify the calculation, first compute the $n \times 1$ vector

$$Z(s) = (sI - A)^{-1}B$$

by examining

$$(sI - A)Z(s) = B$$

that is

$$\begin{bmatrix} s & -1 & & & \\ & & \ddots & & \\ & & & s & -1 \\ a_0 & & \cdots & a_{n-2} & s + a_{n-1} \end{bmatrix} \begin{bmatrix} z_1(s) \\ \vdots \\ z_{n-1}(s) \\ z_n(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

In scalar form this corresponds to the set of equations

$$\begin{gathered} sz_1(s) = z_2(s) \\ sz_2(s) = z_3(s) \\ \vdots \\ sz_{n-1}(s) = z_n(s) \\ a_0 z_1(s) + \cdots + a_{n-2}z_{n-1}(s) + (s + a_{n-1})z_n(s) = 1 \end{gathered}$$

Recursive substitution gives an equation in $z_1(s)$ alone,

$$a_0 z_1(s) + a_1 s z_1(s) + \cdots + a_{n-1}s^{n-1}z_1(s) + s^n z_1(s) = 1$$

Thus

$$z_1(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

and this specifies the other entries:

$$Z(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix}$$

Finally,

$$C(sI-A)^{-1}B = CZ(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

□

This proof provides a recipe for constructing a realization for a proper-rational transfer function. Also, application of partial fraction expansion to (5) and careful consideration of the types of terms that can arise lead to the time-domain version of the realizability result.

Corollary
A unit-impulse response is realizable if and only if it has the *exponential polynomial* form

$$h(t) = \sum_{k=1}^{l} \sum_{j=0}^{m_k-1} h_{kj}\, t^j e^{\lambda_k t} + D\delta(t)$$

where l and $m_1, \ldots, m_l$ are positive integers, and where the following conjugacy constraint is satisfied. If λ_q is complex, then for some $r \neq q$, $\lambda_r = \overline{\lambda}_q$, $m_r = m_q$, and the corresponding coefficients satisfy $h_{rj} = \overline{h}_{qj}$ for $j = 0, \ldots, m_r - 1$.

Considering the general convolution representation for LTI systems, the corollary makes it evident that *realizable* LTI systems form a rather special subclass!

Minimal Realization

Next consider the characterization of minimal realizations of a given (realizable) input-output description, an issue that relies on the concepts introduced in Section 5. The results are phrased in terms of transfer functions, as this representation is more amenable to explicit treatment than is the unit-impulse response. Indeed, the easiest way to write down a realization for a given (realizable) unit-impulse response is to first compute the corresponding transfer function.

Theorem
Suppose (1) is a realization of a given transfer function $H(s)$. Then it is a minimal realization of $H(s)$ if and only if it is controllable and observable.

Proof
We first show that controllability and observability imply minimality by arguing the contrapositive. Suppose (1) is an n-dimensional realization of $H(s)$ that is not minimal. Then there exists a realization of $H(s)$,

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) \\ y(t) &= Hz(t) + Ju(t) \end{aligned} \qquad (6)$$

that has dimension $n_z < n$. Since the unit impulse responses of (1) and (6) are the same, we have $J = D$, and

$$Ce^{At}B = He^{Ft}G\,, \quad t \geq 0$$

Repeated differentiation with respect to t with evaluation at $t = 0$ gives

$$CA^kB = HF^kG\,, \quad k = 0, 1, \ldots$$

We can arrange the first $2n-1$ values of this scalar data into matrix form to obtain

$$\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix} = \begin{bmatrix} HG & HFG & \cdots & HF^{n-1}G \\ HFG & HF^2G & \cdots & HF^nG \\ \vdots & \vdots & \vdots & \vdots \\ HF^{n-1}G & HF^nG & \cdots & HF^{2n-2}G \end{bmatrix}$$

This can be written as

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix}$$

The right side is the product of an $n \times n_z$ matrix and an $n_z \times n$ matrix, and thus cannot have rank greater than n_z. This shows that (1) cannot be both controllable and observable.

Next, suppose that (1) is a minimal, dimension-n, realization of $H(s)$, but that it is not controllable. Then there exists an $1 \times n$, nonzero vector q such that

$$q\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$$

Further, by the Cayley-Hamilton theorem,

$$qA^kB = 0, \quad k = 0,1,\ldots$$

Let P^{-1} be an invertible $n \times n$ matrix with bottom row q, say

$$P^{-1} = \begin{bmatrix} \tilde{P} \\ q \end{bmatrix}$$

and change state variables according to $z(t) = P^{-1}x(t)$ to obtain another minimal realization for $H(s)$:

$$\dot{z}(t) = Fz(t) + Gu(t)$$
$$y(t) = Hz(t) + Du(t)$$

The coefficient matrices in this minimal realization can be written in partitioned form as

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$$

where F_{11} is $(n-1)\times(n-1)$, G_1 is $(n-1)\times 1$, and H_1 is $1\times(n-1)$. The key is the zero at the bottom of G, for further computation gives

$$FG = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11}G_1 \\ F_{21}G_1 \end{bmatrix}$$

and also

$$FG = P^{-1}AP\,P^{-1}B = P^{-1}AB = \begin{bmatrix} \tilde{P}AB \\ qAB \end{bmatrix} = \begin{bmatrix} \tilde{P}AB \\ 0 \end{bmatrix}$$

Thus $F_{21}G_1 = 0$. Continuing yields

$$F^kG = \begin{bmatrix} F_{11}^kG_1 \\ 0 \end{bmatrix}, \quad k = 0,1,\ldots$$

But then

$$\dot{z}(t) = F_{11}z(t) + G_1u(t)$$
$$y(t) = H_1z(t) + Du(t)$$

is a dimension-$(n-1)$ realization for $H(s)$ since, arguing in the time domain,

$$He^{Ft}G = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \sum_{k=0}^{\infty} F^k G \frac{t^k}{k!} = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \sum_{k=0}^{\infty} \begin{bmatrix} F_{11}^k G_1 \\ 0 \end{bmatrix} \frac{t^k}{k!}$$
$$= \sum_{k=0}^{\infty} H_1 F_{11}^k G_1 \frac{t^k}{k!} = H_1 e^{F_{11}t} G_1$$

Of course this contradicts the original minimality assumption, so (1) must be controllable. A similar argument leads to a similar contradiction if we assume that (1) is not observable. Therefore a minimal realization must be both controllable and observable. □

It is natural to refer to a controllable and observable linear state equation as *minimal state equation* in the sense that it is a minimal realization of its own transfer function/unit-impulse response. Another characterization of minimality for a linear state equation that is related to the transfer function viewpoint is the following.

Theorem

The linear state equation (1) is minimal if and only if the polynomials $\det(sI - A)$ and $C\,\text{adj}(sI - A)B$ are coprime (have no roots in common).

Proof

We can assume $D = 0$ since the value of D has no effect on either minimality or on the coprimeness claim. Suppose (1) is minimal and of dimension n. Then its transfer function,

$$H(s) = C(sI - A)^{-1}B = \frac{C\text{adj}(sI - A)B}{\det(sI - A)}$$

is strictly-proper with $\deg\{\det(sI - A)\} = n$. Suppose $\det(sI - A)$ and $C\,\text{adj}(sI - A)B$ have a factor in common. Then by canceling this common factor we can write the transfer function as a strictly-proper rational function of degree no greater than $n-1$. But then the obvious controllability form realization is of this same, lower dimension, which contradicts minimality of (1).

Now suppose that (1) is of dimension n and that the polynomials $\det(sI - A)$ and $C\,\text{adj}(sI - A)B$ are coprime. Writing

$$C\,\text{adj}(sI - A)B = c_{n-1}s^{n-1} + \cdots + c_1 s + c_0$$
$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

another dimension-n realization of the transfer function of (1) is the controllability form state equation

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) \\ y(t) &= Hz(t) \end{aligned} \tag{7}$$

with

$$F = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & & \cdots & & -a_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} c_0 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix}$$

This state equation is controllable, so its minimality – and hence minimality of (1) – is equivalent to observability. Proceeding by contradiction, if (7) is not observable, then there exists an eigenvalue λ of F (a root of $\det(sI - A) = \det(sI - F)$) and corresponding eigenvector p such that

$$Fp = \lambda p\,, \quad Hp = 0$$

In scalar terms,

$$\begin{bmatrix} p_2 \\ \vdots \\ p_n \\ -a_0 p_1 - \cdots - a_{n-1} p_n \end{bmatrix} = \begin{bmatrix} \lambda p_1 \\ \vdots \\ \lambda p_{n-1} \\ \lambda p_n \end{bmatrix}, \quad c_0 p_1 + \cdots + c_{n-1} p_n = 0$$

Using the first equation to express the second in terms of p_1 alone gives

$$(c_0 + c_1 \lambda + \cdots + c_{n-1} \lambda^{n-1}) p_1 = 0$$

Noting that $p_1 \neq 0$, for otherwise $p = 0$, we have that λ also is a root of

$$c_0 + c_1 s + \cdots + c_{n-1} s^{n-1} = C \operatorname{adj}(sI - A) B$$

This contradicts coprimeness, and we conclude that (7) is observable, hence minimal. □

This result implicates common factors in the numerator and denominator of a transfer function as a root cause of non-minimal realizations. In the three-bucket system example of nonminimality, it is easy to verify that

$$\det(sI - A) = (s+1)^3\,, \quad C \operatorname{adj}(sI - A) B = (s+1)^2$$

leading to the transfer function

$$H(s) = C(sI - A)^{-1} B = \frac{C \operatorname{adj}(sI - A) B}{\det(sI - A)} = \frac{(s+1)^2}{(s+1)^3}$$

$$= \frac{1}{s+1}$$

Another easy consequence is that for a minimal, dimension-n state equation, the set of n poles (including multiplicities) of its transfer function is identical to the set of n eigenvalues of A. Finally, a method for constructing a minimal realization for a given (realizable) transfer function is to first isolate the D as in (3), then cancel any common factors from the numerator and denominator of the strictly-proper rational portion, and then write, by inspection, the controllability form A, B, and C as in (4).

Though realizations of an input-output description are highly non-unique, for minimal realizations the non-uniqueness is essentially in the choice of internal (state) variables.

Theorem

Suppose (1) and

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) \\ y(t) &= Hz(t) + Ju(t) \end{aligned} \tag{8}$$

are both n-dimensional minimal realizations of a given transfer function $H(s)$. Then $J = D$ and there exists a unique, invertible, $n \times n$ matrix P such that

$$F = P^{-1} A P\,, \quad G = P^{-1} B\,, \quad H = CP \tag{9}$$

Proof

Since $J = D$ is obvious, we focus on the construction of P such that (9) holds. By hypothesis,

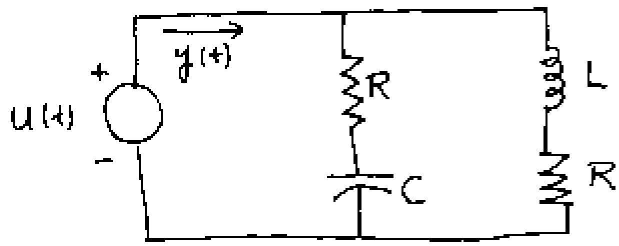
$$Ce^{At} B = He^{Ft} G\,, \quad t \geq 0$$

Differentiating repeatedly with respect to t and evaluating at $t = 0$ gives

$$CA^kB = HF^kG\,, \quad k = 0,1,\ldots \tag{10}$$
Writing the controllability and observability matrices, all invertible, for the two realizations as
$$C_A = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, \quad C_F = \begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix}$$
$$O_A = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad O_F = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$
the scalar equalities can be arranged in matrix form to yield
$$O_AC_A = O_FC_F \tag{11}$$
Now set $P = C_AC_F^{-1}$, which by (11) is the same as $P = O_A^{-1}O_F$. Then
$$P^{-1} = C_FC_A^{-1} = O_F^{-1}O_A$$
and (9) is verified as follows. We can write
$$C_F = C_FC_A^{-1}C_A = P^{-1}C_A$$
the first column of which gives $G = P^{-1}B$. Similarly,
$$O_F = O_AO_A^{-1}O_F = O_AP \tag{12}$$
the first row of which gives $H = CP$. Finally, the scalar equalities in (10) also can be arranged into the matrix equality
$$O_AAC_A = O_FFC_F$$
This permits the calculation
$$P^{-1}AP = O_F^{-1}O_AAC_AC_F^{-1} = O_F^{-1}O_FFC_FC_F^{-1} = F$$
To show uniqueness of P , suppose Q is such that
$$F = Q^{-1}AQ\,, \quad G = Q^{-1}B\,, \quad H = CQ$$
Then
$$HF^k = CQ(Q^{-1}AQ)^k = CA^kQ\,, \quad k = 0,1,\ldots$$
and thus
$$O_F = O_AQ \tag{13}$$
From (12) and (13) we have
$$O_A(P-Q) = 0$$
and invertibility of O_A gives $P = Q$. □

Exercises

1. For the electrical circuit shown below, with voltage input $u(t)$ and current output $y(t)$, compute the transfer function of the circuit (the *driving-point admittance*). What is the dimension of minimal realizations of this transfer function? What happens if $R^2C = L$? Explain.



2. For the transfer function

$$H(s) = \frac{(-1/4)s^2}{(s+2)^2}$$

provide realizations that are controllable and observable, controllable but not observable, observable but not controllable, and neither controllable nor observable.

3. Show that a linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is minimal if and only if

$$\dot{x}(t) = (A - BC)x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is minimal. What is the relationship between the two state equations?

4. For what values of the parameter α is the linear state equation

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & \alpha & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t)$$

minimal?

5. Given any $n \times n$ matrix *A*, do there exist $n \times 1$ *B* and $1 \times n$ *C* such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is minimal?

6. Consider the cascade connection of two minimal linear state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ w(t) &= Cx(t) + Du(t) \end{aligned} \quad \text{and} \quad \begin{aligned} \dot{z}(t) &= Fz(t) + Gv(t) \\ y(t) &= Hz(t) + Jv(t) \end{aligned}$$

(That is, set $v(t) = w(t)$.) Write a linear state equation for the overall system and establish a necessary and sufficient condition for it to be minimal.

7. Stability Again

With the tools of controllability and observability available, we return to the stability issue left unresolved in Section 4, namely, the relationship between asymptotic stability and uniform bounded-input, bounded-output stability for the linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

Equivalence of External and Internal Stability

The first result is straightforward, and likely recognized by astute readers of Section 4.

Theorem
If the linear state equation (1) is asymptotically stable, then it is uniformly bounded-input, bounded-output stable.

Proof
Using the representation for the matrix exponential derived from partial fraction expansion, we can write

$$Ce^{At}B = \sum_{k=1}^{l}\sum_{j=1}^{m_k} CW_{kj}B\frac{t^{j-1}}{(j-1)!}e^{\lambda_k t} \tag{2}$$

where each λ_k is an eigenvalue of A with multiplicity m_k. Since each eigenvalue has negative real part,

$$\int_0^\infty \left|Ce^{At}B\right| dt \le \sum_{k=1}^{l}\sum_{j=1}^{m_k} \left|CW_{kj}B\right| \frac{1}{(j-1)!}\int_0^\infty t^{j-1}e^{\lambda_k t}\,dt$$

and the right side is finite. Thus the state equation is uniformly bounded-input, bounded-output stable. □

It is the converse that requires the developments in Sections 5 and 6.

Theorem
If the linear state equation (1) is controllable, observable, and uniformly bounded-input, bounded-output stable, then it is asymptotically stable.

Proof
Since (1) is uniformly bounded-input, bounded-output stable, the absolute integrability of the unit-impulse response implies

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} Ce^{At}B = 0$$

Moreover, since $h(t)$ has the exponential polynomial form (2), it is easy to see that that the time derivative of $h(t)$ also has exponential polynomial form, with the same λ_k's in the exponents but different coefficients. Therefore

$$\lim_{t\to\infty} \dot{h}(t) = \lim_{t\to\infty} CAe^{At}B = \lim_{t\to\infty} Ce^{At}AB = 0$$

This argument can be repeated for subsequent derivatives to give

$$\lim_{t\to\infty} \ddot{h}(t) = \lim_{t\to\infty} CA^2 e^{At} B = \lim_{t\to\infty} CAe^{At} AB = \lim_{t\to\infty} Ce^{At} A^2 B = 0$$

$$\vdots$$

$$\begin{aligned}\lim_{t\to\infty} h^{(2n-2)}(t) &= \lim_{t\to\infty} CA^{2n-2} e^{At} B = \lim_{t\to\infty} CA^{2n-3} e^{At} AB \\ &= \cdots = \lim_{t\to\infty} CAe^{At} A^{2n-3} B = \lim_{t\to\infty} Ce^{At} A^{2n-2} B \\ &= 0\end{aligned}$$

Arranging this data in an $n \times n$ matrix form gives

$$\begin{aligned}0 &= \lim_{t\to\infty} \begin{bmatrix} Ce^{At}B & Ce^{At}AB & \cdots & Ce^{At}A^{n-1}B \\ CAe^{At}B & CAe^{At}AB & \cdots & CAe^{At}A^{-1}B \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-1}e^{At}B & CA^{n-1}e^{At}AB & \cdots & CA^{n-1}e^{At}A^{n-1}B \end{bmatrix} \\ &= \lim_{t\to\infty} \left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} e^{At} \begin{bmatrix} B & AB & \cdots & A^{-1}B \end{bmatrix} \right) \\ &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \left(\lim_{t\to\infty} e^{At} \right) \begin{bmatrix} B & AB & \cdots & A^{-1}B \end{bmatrix}\end{aligned}$$

Using the controllability and observability assumptions, we can multiply this expression by the appropriate inverses to conclude that

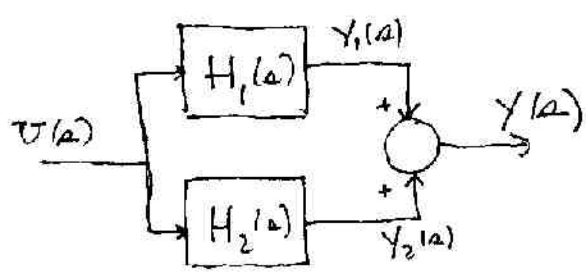
$$\lim_{t\to\infty} e^{At} = 0$$

which implies asymptotic stability of (1). □

Thus the hidden instability issue in Section 4 cannot arise for a minimal linear state equation.

Stability of Interconnected Systems

Using these results we can address issues of hidden stability for interconnections of LTI systems, always assuming that the subsystems are described by minimal (controllable and observable) realizations so that the internal and external stability properties of the subsystems are equivalent. For example, consider the additive parallel connection shown below



where the subsystem $H_1(s)$ is described by the minimal state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y_1(t) &= Cx(t) + Du(t)\end{aligned}$$

and $H_2(s)$ is described by the minimal state equation

$$\dot{z}(t) = Fz(t) + Gu(t)$$
$$y_2(t) = Hz(t) + Ju(t)$$

The overall system is described by the transfer function

$$H(s) = H_1(s) + H_2(s) \tag{3}$$

and by the linear state equation

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + (D+J)u(t) \end{aligned} \tag{4}$$

Theorem
For the additive-parallel connection, the overall system (4)
(a) is asymptotically stable if and only if each subsystem is asymptotically stable,
(b) is uniformly bounded-input, bounded-output stable if each subsystem is uniformly bounded-input, bounded-output stable.

Proof
Claim (a) follows from the fact that the eigenvalues of

$$\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}$$

are given by the union of the eigenvalues of A and the eigenvalues of F. Claim (b) is equally straightforward, and follows by noting from (3), written over the common denominator, that the poles of $H(s)$ are a subset of the union of the poles of $H_1(s)$ and the poles of $H_2(s)$.
□

Remark
This result obviously applies to additive-parallel connections of any number of LTI subsystems. Also, an easy counterexample to the converse of claim *(b)* is provided by

$$H_1(s) = \frac{2s}{s^2-1} = \frac{1}{s+1} + \frac{1}{s-1}$$
$$H_2(s) = \frac{-3}{s^2+s-2} = \frac{1}{s+2} - \frac{1}{s-1}$$

and corresponding minimal realizations.
□

Analogous results for the series connection of two LTI systems are the subject of Exercise 7.1. The feedback connection is discussed in subsequent sections, as it is much more subtle. (Indeed, the feedback connection of LTI systems may not even be well defined!)

Exercises

1. Consider the series connection of two minimal state equations (subsystems)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ w(t) &= Cx(t) + Du(t) \end{aligned} \quad \text{and} \quad \begin{aligned} \dot{z}(t) &= Fz(t) + Gv(t) \\ y(t) &= Hz(t) + Jv(t) \end{aligned}$$

(That is, set $v(t) = w(t)$.) Provide proofs or counterexamples to the following claims.

(a) If the two subsystem state equations are asymptotically stable (respectively, uniformly bounded-input, bounded-output stable), then the overall system is asymptotically stable (respectively, uniformly bounded-input, bounded-output stable).

(b) If the overall system is asymptotically stable (respectively, uniformly bounded-input, bounded-output stable), then the two subsystems are asymptotically stable (respectively, uniformly bounded-input, bounded-output stable).

2. Show that the SISO state equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + u(t)$$

and

$$\dot{x}(t) = (A - BC)x(t) + Bu(t)$$
$$y(t) = -Cx(t) + u(t)$$

are inverses of each other in the sense that the product of their transfer functions is unity. If the first state equation is uniformly bounded-input, bounded-output stable, is the second state equation?

8. LTI Feedback

In this section we begin the study of feedback in LTI systems from the viewpoints of both state equation representations and transfer function representations.

State and Output Feedback

Consider the dimension-n linear state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

often called the *open-loop* state equation in this context, and the linear state feedback

$$u(t) = Kx(t) + Nr(t) \tag{2}$$

where K is $1 \times n$ and N is a scalar. This yields the *closed-loop* state equation

$$\begin{aligned}\dot{x}(t) &= (A + BK)x(t) + BNr(t) \\ y(t) &= (C + DK)x(t) + DNr(t)\end{aligned} \tag{3}$$

In a similar manner, we can consider linear output feedback

$$u(t) = Ly(t) + Nr(t) \tag{4}$$

where L and N both are scalars. But, in this case an issue arises when (4) is expressed in terms of the state,

$$u(t) = LCx(t) + LDu(t) + Nr(t) \tag{5}$$

Namely, when $LD = 1$ the input $u(t)$ is undefined. The following terminology is standard.

Definition
The output feedback system (1), (4) is said to be *well posed* if $LD \neq 1$.

When the output feedback system is well posed, substitution of (5) into (1) yields the closed-loop state equation

$$\begin{aligned}\dot{x}(t) &= (A + BLC\tfrac{1}{1-LD})x(t) + BN\tfrac{1}{1-LD}r(t) \\ y(t) &= C\tfrac{1}{1-LD}x(t) + \tfrac{1}{1-LD}DNr(t)\end{aligned} \tag{6}$$

Basic properties describing the zero-state and zero-input responses of the closed-loop state equation in terms of the open-loop state equation are provided by the following results. Note that closed-loop quantities can be written in explicit terms of open-loop quantities only when the Laplace transform representation is adopted.

Theorem
For the linear state equation (1) with state feedback (2), the open and closed-loop matrix exponentials are related by

$$e^{(A+BK)t} = e^{At} + \int_0^t e^{A(t-\sigma)}BKe^{(A+BK)\sigma}d\sigma \tag{7}$$

In terms of Laplace transforms,

$$(sI - A - BK)^{-1} = \left[I - (sI - A)^{-1}BK\right]^{-1}(sI - A)^{-1} \tag{8}$$

With output feedback (4), assuming $LD \neq 1$, these relations hold after replacing K by

$$\frac{L}{1 - LD}C$$

Proof
Writing the right side of (7) as

$$F(t) = e^{At} + e^{At}\int_0^t e^{-A\sigma} BKe^{(A+BK)\sigma} d\sigma$$

we have

$$\begin{aligned}\dot{F}(t) &= Ae^{At} + Ae^{At}\int_0^t e^{-A\sigma} BKe^{(A+BK)\sigma} d\sigma + e^{At}\, e^{-At} BKe^{(A+BK)t} \\ &= AF(t) + BKe^{(A+BK)t}\end{aligned} \tag{9}$$

and, furthermore, $F(0) = I$. But $e^{(A+BK)t}$ is the unique solution of

$$\dot{F}(t) = (A+BK)F(t) = AF(t) + BKF(t)\,, \quad F(0) = I$$

Therefore

$$\dot{F}(t) = AF(t) + BKe^{(A+BK)t}\,, \quad F(0) = I$$

implies $F(t) = e^{(A+BK)t}$, and (7) is verified. Taking the Laplace transform of (7), using in particular the convolution property, gives

$$(sI - A - BK)^{-1} = (sI - A)^{-1} + (sI - A)^{-1} BK(sI - A - BK)^{-1}$$

an expression that easily rearranges to (8). The corresponding results for a well-posed output feedback system are obvious.
□

For the zero-state response, a useful expression relating the open- and closed-loop unit-impulse responses or transfer functions is not available for the case of state feedback. However the situation is somewhat better for output feedback, assuming it is well posed.

Theorem
Consider the linear state equation (1) with output feedback (4), where $LD \neq 1$. Then the open- and closed-loop unit-impulse responses (in an obvious notation) are related by

$$h_{cl}(t) = Nh_{ol}(t) + L\int_0^t h_{ol}(t-\sigma)h_{cl}(\sigma)d\sigma\,, \quad t \geq 0 \tag{10}$$

In terms of Laplace transforms

$$H_{cl}(s) = \frac{NH_{ol}(s)}{1 - LH_{ol}(s)} \tag{11}$$

Proof
To simplify notation, let

$$\alpha = \frac{1}{1 - LD}$$

Then with $K = \alpha LC$, (7) gives

$$e^{(A+B\alpha LC)t} = e^{At} + \int_0^t e^{A(t-\sigma)} B\alpha LCe^{(A+B\alpha LC)\sigma} d\sigma$$

Multiplying by $\alpha^2 NC$ on the left, and B on the right yields

$$\alpha^2 N\, Ce^{(A+B\alpha LC)t} B = \alpha^2 N\, Ce^{At} B + \int_0^t \alpha Ce^{A(t-\sigma)} \alpha NB\, L\, \alpha Ce^{(A+B\alpha LC)\sigma} \alpha NBd\sigma \tag{12}$$

From (1) and (6) we can write the open- and closed-loop unit-impulse responses as

$$h_{ol}(t) = Ce^{At}B + D\delta(t)$$
$$h_{cl}(t) = \alpha Ce^{(A+\alpha LBC)t}\alpha NB + \alpha DN\delta(t)$$

Substituting into (12) and evaluating the impulsive convolutions gives (10). Finally, the Laplace transform of (10) , again using the convolution property, is

$$H_{cl}(s) = NH_{ol}(s) + LH_{ol}(s)H_{cl}(s)$$

an expression that yields (11). □

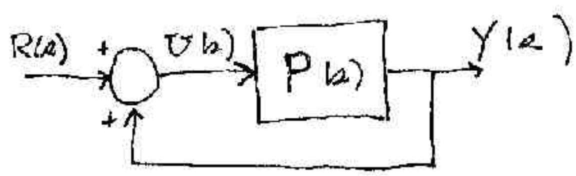
Obviously these results are for *static* LTI feedback. State equation analysis for feedback involving dynamic LTI systems is discussed in the sequel.

Transfer Function Analysis

Feedback often is addressed in terms of transfer function representations, for reasons that should be clear from the results above, and again issues arise about whether a feedback interconnection of LTI systems is well defined. We assume throughout that the subsystem transfer functions are proper rational functions. (Entrenched notation motivates a change of symbolism.)

Example

For the simple unity-output-feedback system shown



with

$$P(s) = \frac{s-1}{s+2}$$

application of (11) gives the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{P(s)}{1-P(s)} = \frac{\frac{s-1}{s+2}}{1-\frac{s-1}{s+2}} = \frac{s-1}{3} \qquad (13)$$

which of course is not a proper rational function. Also the transfer function

$$\frac{U(s)}{R(s)} = \frac{1}{1-P(s)} = \frac{s+2}{3}$$

is improper. This actually is a more basic observation, since (13) is obtained by multiplication by a proper rational function:

$$\frac{Y(s)}{R(s)} = \frac{Y(s)}{U(s)}\frac{U(s)}{R(s)} = P(s)\frac{U(s)}{R(s)}$$

In any case, the situation is easily seen to be a manifestation of the failure of the well-posed condition for output feedback in a state equation setting by considering a (minimal) realization for $P(s)$,

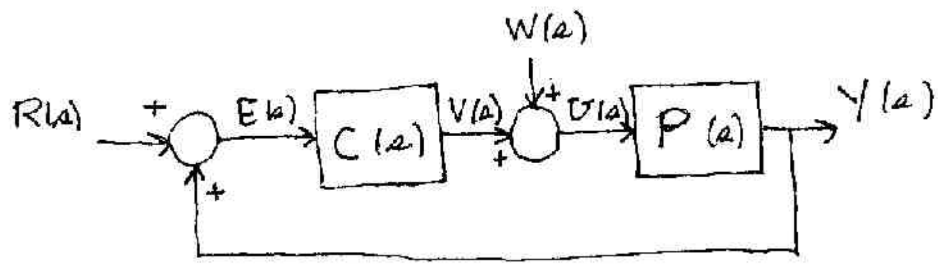
$$\dot{x}(t) = -2x(t) + u(t)$$
$$y(t) = -3x(t) + u(t)$$

and writing the feedback as

$$u(t) = y(t) + r(t) = -3x(t) + u(t) + r(t)$$

That is, the closed-loop system cannot be described by a linear state equation (1), for such a state equation cannot have an improper transfer function as in (13).
□

In addition to the fact that improper rational functions are outside the class of LTI systems we consider, Exercise 4.9 provides further motivation for avoiding them. We use the following terminology in the transfer function setting. An interconnection of (proper rational) transfer functions is called *well posed* if all transfer functions from external inputs to internal signals in the closed-loop system are proper. Details and further specificity depend on the particular interconnection at hand, and a general analysis that covers a wide range of interconnections is rather complex. For simplicity, we will focus on the unity feedback system shown below, where the external inputs signals are a reference input $R(s)$ and a disturbance input $W(s)$. Of course we assume that the subsystem transfer functions, $C(s)$ and $P(s)$, are proper-rational transfer functions.



From the block diagram,

$$E(s) = R(s) + Y(s) = R(s) + P(s)(V(s) + W(s))$$
$$= R(s) + P(s)C(s)E(s) + P(s)W(s)$$

which gives

$$E(s) = \frac{1}{1 - P(s)C(s)} R(s) + \frac{P(s)}{1 - P(s)C(s)} W(s) \tag{14}$$

Also

$$U(s) = W(s) + V(s) = W(s) + C(s)\big(R(s) + Y(s)\big)$$
$$= W(s) + C(s)R(s) + P(s)C(s)U(s)$$

gives

$$U(s) = \frac{C(s)}{1 - P(s)C(s)} R(s) + \frac{1}{1 - P(s)C(s)} W(s) \tag{15}$$

Thus the feedback system is well posed if (and only if) the three distinct transfer functions occurring in (14) and (15) are proper. (Note that transfer functions from $R(s)$ and $W(s)$ to $V(s)$ and $Y(s)$ are products of proper rational (subsystem) transfer functions and the transfer functions in (14) and (15).) However, it is clear that only one of the transfer functions need be checked, namely

$$\frac{1}{1 - P(s)C(s)}$$

since products of proper rational functions are proper rational functions.

It is traditional in the transfer function setting to explicitly display the "D-terms" in $C(s)$ and $P(s)$ by writing

$$\begin{aligned} C(s) &= C(\infty) + C_{sp}(s) \\ P(s) &= P(\infty) + P_{sp}(s) \end{aligned} \tag{16}$$

where $C_{sp}(s)$ and $P_{sp}(s)$ are strictly proper rational functions with monic denominator polynomials. That is, by evaluating a proper rational function with denominator degree n as $|s| \to \infty$, the coefficient of s^n in the numerator polynomial is obtained.

Theorem

If $C(s)$ and $P(s)$ are proper rational transfer functions, then the unity feedback system is well posed if and only if $1 - C(\infty)P(\infty) \neq 0$.

Proof

We need only show that

$$\frac{1}{1 - C(s)P(s)} \tag{17}$$

is proper if and only if $1 - C(\infty)P(\infty) \neq 0$. But writing

$$1 - C(s)P(s) = \left(1 - C(\infty)P(\infty)\right) + \left(P(\infty)C_{sp}(s) + P_{sp}(s)C(\infty) + P_{sp}(s)C_{sp}(s)\right)$$

which is the sum of a constant and a strictly-proper rational function, it follows that (17) is proper if and only if the constant, $1 - C(\infty)P(\infty)$, is nonzero.

□

An obvious consequence that applies to the typical situation in beginning courses on feedback control, where the notion of well-posed feedback systems usually is not addressed, is that the feedback system is well posed if *either* $C(s)$ or $P(s)$ is strictly proper.

Exercise 8.1 deals with a state equation analysis of the unity-feedback system.

Exercises

*1***.** For the unity feedback system, suppose that the subsystem $P(s)$ is described by the minimal linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

and $C(s)$ is described by the minimal state equation

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Ge(t) \\ v(t) &= Hz(t) + Je(t) \end{aligned}$$

Using the relationships

$$e(t) = r(t) + y(t)\,, \quad u(t) = v(t) + w(t)$$

to write the closed-loop state equation in terms of the state vector

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

with inputs $r(t)$ and $w(t)$, and output $y(t)$, derive a necessary and sufficient condition for the system to be well posed.

*2***.** Consider the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = C_1 x(t) + Du(t)$$
Suppose $r(t)$ is a reference input signal, and the vector signal
$$v(t) = C_2 x(t) + E_1 r(t) + E_2 u(t)$$
is available for feedback. For the n_c-dimensional dynamic feedback
$$\dot{z}(t) = Fz(t) + Gv(t)$$
$$u(t) = Hz(t) + Jv(t)$$
compute, under appropriate assumptions, the coefficient matrices for the $(n+n_c)-$dimensional closed-loop state equation.

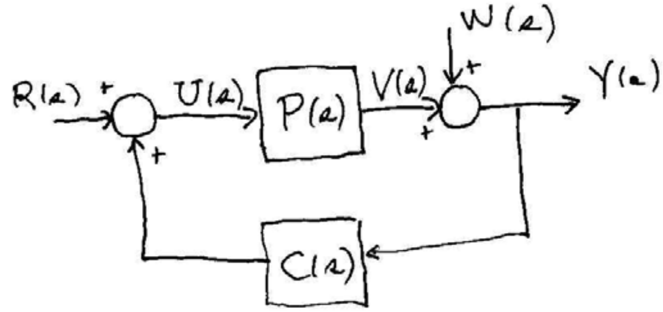
*3***.** The SISO state equation
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$
is said to have *unity dc-gain* if for any given constant $\tilde{u}$ there exists an $n \times 1$ vector $\tilde{x}$ such that
$$A\tilde{x} + B\tilde{u} = 0\,, \quad C\tilde{x} = \tilde{u}$$
Under the assumption that
$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$
is invertible, show that

(a) if a $1 \times n$ matrix K is such that $(A+BK)$ is invertible, then $C(A+BK)^{-1}B$ is nonzero

(b) if a $1 \times n$ matrix K is such that $(A+BK)$ is invertible, then there exists a constant gain G such that the state equation
$$\dot{x}(t) = (A+BK)x(t) + BGu(t)$$
$$y(t) = Cx(t)$$
has identity dc-gain. *Hint:* Work with the matrix
$$\begin{bmatrix} A+BK & B \\ C & 0 \end{bmatrix}$$
and its inverse.

*4***.** For the linear state equation
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$
with state feedback
$$u(t) = Kx(t) + Nr(t)$$
show that the transfer function of the closed-loop state equation can be written in terms of the open-loop transfer function as
$$C(sI-A)^{-1}B[I-K(sI-A)^{-1}B]^{-1}N$$
(This shows that the input-output behavior of the closed-loop system can be obtained by using a precompensator instead of feedback.) *Hint:* Verify the following identity for an $n \times m$ matrix *P* and an $m \times n$ matrix *Q*, where the indicated inverses are assumed to exist:
$$P(I-QP)^{-1} = (I-PQ)^{-1}P$$

5. For the feedback system shown below, where $P(s)$ and $C(s)$ are proper rational functions,



derive a necessary and sufficient condition for the system to be well posed.

9. Feedback Stabilization

In this section we discuss the use of feedback to stabilize a given LTI open-loop state equation or transfer function. Attention is focused on obtaining an internally stable (asymptotically stable) closed-loop system, for this implies uniform bounded-input, bounded-output stability, and avoids issues of hidden instability discussed in Sections 4 and 7. Basic necessary and sufficient conditions are provided in the case of static state feedback, and a characterization of all stabilizing controllers is provided in the transfer function setting. In the course of our development, the celebrated eigenvalue assignability result for static state feedback is established.

State Feedback Stabilization

Consider the dimension-n linear state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

with linear state feedback

$$u(t) = Kx(t) + Nr(t) \tag{2}$$

where K is $1 \times n$ and N is a scalar. This yields the closed-loop state equation

$$\begin{aligned}\dot{x}(t) &= (A + BK)x(t) + BNr(t) \\ y(t) &= (C + DK)x(t) + DNr(t)\end{aligned} \tag{3}$$

Focusing on asymptotic stability of (3), we can ignore the output equation and the input term. The issue is existence of K such that all eigenvalues of $A + BK$ have negative real parts.

Remark

Static *output* feedback might be proposed as a more convenient mechanism for obtaining an asymptotically stable closed-loop state equation, but easy examples show that this approach often fails. Moreover, it turns out to be difficult to delineate properties of the open-loop state equation that are equivalent to existence of a stabilizing (static) output feedback. The case of dynamic output feedback in the state equation setting is addressed in the sequel. □

It is obvious from intuition, or simple examples with diagonal A, that the concept of controllability, or something like it, is involved. Indeed, we approach the issue by first establishing the result that controllability is *sufficient* for stabilization, and then we develop a necessary and sufficient condition. Since eigenvalues of a real matrix must occur in complex conjugate pairs, this constraint often is left understood, and the following result is referred to as the *eigenvalue assignability* theorem.

Theorem

Suppose that (1) is controllable. Then given any set of n complex numbers, $\lambda_1, \ldots, \lambda_n$, conjugates included, there exists a state feedback gain K such that these are the eigenvalues of $A + BK$.

Proof

Since (1) is controllable, from Section 5 we can compute an invertible, $n \times n$ matrix P to obtain the controllability form coefficients

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & & \cdots & & -a_{n-1} \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where the bottom row of $P^{-1}AP$ displays the coefficients of the characteristic polynomial of A. Given a desired set of eigenvalues, $\lambda_1,\ldots,\lambda_n$, let $p(\lambda)$ be the corresponding desired characteristic polynomial,

$$\begin{aligned} p(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \\ &= \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0 \end{aligned}$$

Letting

$$K_{cf} = \begin{bmatrix} -p_0 + a_0 & -p_1 + a_1 & \cdots & -p_{n-1} + a_{n-1} \end{bmatrix}$$

an easy calculation gives

$$P^{-1}AP + P^{-1}BK_{cf} = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -p_0 & & \cdots & & -p_{n-1} \end{bmatrix}$$

which shows that controllability form is preserved under state feedback. Moreover, the coefficients of the bottom row show that this matrix has the desired eigenvalues. Since

$$P^{-1}AP + P^{-1}BK_{cf} = P^{-1}\left(A + BK_{cf}P^{-1}\right)P$$

and a similarity transformation does not alter eigenvalues it is clear that the gain $K = K_{cf}P^{-1}$ is such that $A + BK$ has eigenvalues $\lambda_1,\ldots,\lambda_n$.

□

Obviously controllability is a sufficient condition for existence of a stabilizing state feedback gain, simply require that the desired eigenvalues have negative real parts, but a sharper condition can be proved.

Theorem
For the linear state equation (1), there exists a state feedback gain K such that all eigenvalues of $A + BK$ have negative real parts if and only if

$$rank\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \tag{4}$$

for every λ that is a *nonnegative*-real-part eigenvalue of A.

Proof
From a lemma in Section 5, supposing that

$$rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = q$$

where $0 < q < n$ (the cases $q = 0$ and $q = n$ are trivial) there exists an invertible, $n \times n$ matrix P such that

$$F = P^{-1}AP = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \quad G = P^{-1}B = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{5}$$

where F_{11} is $q \times q$, G_1 is $q \times 1$, and

$$\text{rank}\begin{bmatrix} G_1 & F_{11}G_1 & \cdots & F_{11}^{q-1}G_1 \end{bmatrix} = q$$

Equivalently,

$$\text{rank}\begin{bmatrix} \lambda I - F_{11} & G_1 \end{bmatrix} = q$$

for all complex values of λ. Then, from (5), the eigenvalues of A comprise the eigenvalues of F_{11} and F_{22}, and for any complex value of λ,

$$rank\begin{bmatrix}\lambda I - F & G\end{bmatrix} = rank\begin{bmatrix}\lambda I - F_{11} & -F_{12} & G_1 \\ 0 & \lambda I - F_{22} & 0\end{bmatrix} = q + rank\begin{bmatrix}\lambda I - F_{22}\end{bmatrix} \qquad (6)$$

Now suppose that (4) holds for every nonnegative-real-part eigenvalue of A. Then from Exercise 9.xx,

$$rank\begin{bmatrix}\lambda I - F & G\end{bmatrix} = n$$

for every nonnegative-real-part eigenvalue of F , and (6) shows that all eigenvalues of F_{22} must have negative real parts. Using the eigenvalue assignability theorem on the controllable subsystem, we can compute $1\times q$ K_1 such that all eigenvalues of $F_{11} + G_1 K_1$ have negative real parts. Then setting

$$K = \begin{bmatrix}K_1 & 0\end{bmatrix}$$

we have that

$$F + GK = \begin{bmatrix}F_{11} + G_1 K_1 & F_{12} \\ 0 & F_{22}\end{bmatrix}$$

has negative real part eigenvalues. As in the proof above, this implies that the gain KP^{-1} is such that $(A + BKP^{-1})$ has negative real parts, so (1) is stabilizable.
Now suppose (1) is stabilizable. Going through the change of variables again, there exists a $K = \begin{bmatrix}K_1 & K_2\end{bmatrix}$ such that

$$F + GK = \begin{bmatrix}F_{11} + G_1 K_1 & F_{12} + G_1 K_2 \\ 0 & F_{22}\end{bmatrix}$$

has negative-real-part eigenvalues, which implies that F_{22} has negative-real-part eigenvalues. Thus for any λ with nonnegative real part, (6) implies that

$$rank\begin{bmatrix}\lambda I - F & G\end{bmatrix} = q + n - q = n$$

Invoking Exercise 9.xx once more shows that (4) holds for any λ with nonnegative real part
□

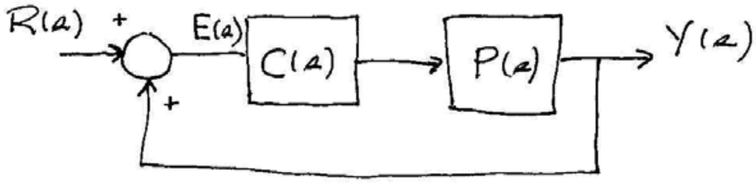
For apparent reasons, the rank condition in this result referred to as the *stabilizability* condition.

Transfer Function Analysis

Basic issues of stabilization are somewhat different when feedback is treated in a transfer function setting, because feedback occurs via subsystem outputs, possibly through *compensators* (additional LTI subsystems). We will consider the issue for unity feedback systems of the type treated in Section 8, leaving other cases to the exercises. Throughout we assume that each subsystem is described by a coprime, proper-rational transfer function, and that the feedback system is well posed.

The subtlety of the stabilization issue can be illustrated by a very simple case.

Example
For the system shown below



with

$$C(s)=\frac{s-1}{s+1},\quad P(s)=\frac{-1}{s-1}$$

a simple calculation gives

$$\frac{Y(s)}{R(s)}=\frac{C(s)P(s)}{1-C(s)P(s)}=\frac{-1}{s+2}$$

Clearly the closed-loop system is uniformly bounded-input, bounded-output stable. However, it is not internally stable, as the following state equation analysis reveals. Taking the minimal realization

$$\begin{aligned}\dot{x}(t)&=x(t)+v(t)\\ y(t)&=-x(t)\end{aligned}$$

for $P(s)$, and

$$\begin{aligned}\dot{z}(t)&=-z(t)+e(t)\\ v(t)&=-2z(t)+e(t)\end{aligned}$$

for $C(s)$, a closed-loop state equation can be formed by using the relation $e(t)=r(t)+y(t)=r(t)-x(t)$ to eliminate the variable $e(t)$. This gives

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix}=\begin{bmatrix}0 & -2\\ -1 & -1\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}r(t)$$

$$y(t)=\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}$$

From calculation of the characteristic polynomial of the closed-loop “A” matrix,

$$\det\begin{bmatrix}\lambda & 2\\ 1 & \lambda+1\end{bmatrix}=\lambda^2+\lambda-2=(\lambda-1)(\lambda+2)$$

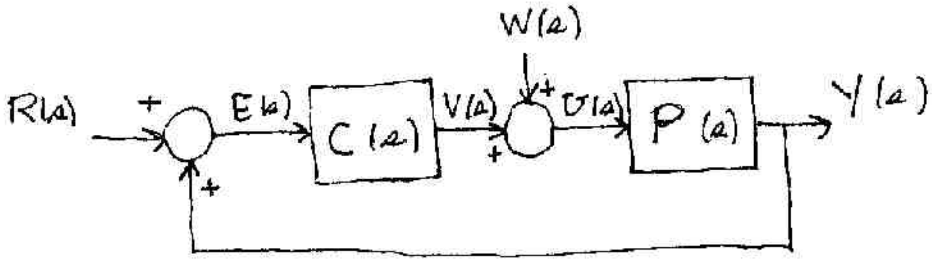
we see a positive eigenvalue which implies that the closed-loop state equation is not asymptotically stable.

□

To capture the notion of internal stability from a transfer function perspective is not completely trivial. In the example the additional transfer functions

$$\frac{E(s)}{R(s)},\quad \frac{V(s)}{R(s)}$$

also have negative-real-part poles, so the instability is hidden from these as well. The key is to add additional input and output signals such that the internal instability appears as a failure of input-output stability in at least one of the closed-loop transfer functions from an input to an output. In the unity feedback system, if we add an input $W(s)$, shown below



then

$$\frac{Y(s)}{W(s)} = \frac{P(s)}{1 - C(s)P(s)} = \frac{-(s+1)}{(s-1)(s+2)}$$

is not uniformly bounded-input, bounded-output stable. Similar examples show that we must also define another output signal in order to avoid other hidden-instability possibilities.

The analysis must be performed for the particular interconnection structure at hand. We will consider the system shown above, with inputs $R(s)$ and $W(s)$, and outputs $Y(s)$ and $V(s)$. It is assumed that $C(s)$ and $P(s)$ are coprime, proper rational transfer functions, and that the feedback system is well posed. It is straightforward to calculate the transfer functions,

$$Y(s) = \frac{P(s)C(s)}{1 - P(s)C(s)} R(s) + \frac{P(s)}{1 - P(s)C(s)} W(s) \tag{7}$$

and

$$V(s) = \frac{C(s)}{1 - P(s)C(s)} R(s) + \frac{P(s)C(s)}{1 - P(s)C(s)} W(s) \tag{8}$$

Lemma
If all poles of the transfer functions

$$\frac{Y(s)}{W(s)} = \frac{P(s)}{1 - P(s)C(s)} \quad , \quad \frac{V(s)}{R(s)} = \frac{C(s)}{1 - P(s)C(s)} \tag{9}$$

have negative real parts, then all poles of

$$\frac{P(s)C(s)}{1 - P(s)C(s)}$$

have negative real parts.

Proof
Write the coprime subsystem transfer functions to display the numerator and denominator polynomials as

$$P(s) = \frac{n_P(s)}{d_P(s)}, \quad C(s) = \frac{n_C(s)}{d_C(s)}$$

Then straightforward calculations give

$$\frac{P(s)}{1-P(s)C(s)}=\frac{n_P(s)d_C(s)}{d_P(s)d_C(s)-n_P(s)n_C(s)}$$
$$\frac{C(s)}{1-P(s)C(s)}=\frac{n_C(s)d_P(s)}{d_P(s)d_C(s)-n_P(s)n_C(s)}$$
$$\frac{P(s)C(s)}{1-P(s)C(s)}=\frac{n_P(s)n_C(s)}{d_P(s)d_C(s)-n_P(s)n_C(s)}$$

Suppose that the first two transfer functions have negative-real-part poles, but the third has a pole s_o with $\mathrm{Re}\{s_o\}\geq 0$. Then

$$d_P(s_o)d_C(s_o)-n_P(s_o)n_C(s_o)=0 \tag{10}$$

and

$$\begin{aligned} n_C(s_o)d_P(s_o)&=0\\ n_P(s_o)d_C(s_o)&=0 \end{aligned} \tag{11}$$

Therefore either $n_C(s_o)=0$ or $d_P(s_o)=0$ and either $n_P(s_o)=0$ or $d_C(s_o)=0$. Due to coprimeness, $n_P(s_o)$ and $d_P(s_o)$ cannot both be zero, nor can $n_C(s_o)$ and $d_C(s_o)$. This implies that either $n_C(s_o)=n_P(s_o)=0$, which by (10) implies $d_P(s_o)d_C(s_o)=0$, a contradiction of subsystem coprimeness, or $d_C(s_o)=d_P(s_o)=0$, which by (10) implies $n_P(s_o)n_C(s_o)=0$, another contradiction.

□

Theorem

The well-posed unity feedback system is internally stable if and only if all poles of the transfer functions in (9) have negative real parts.

Proof

To address internal stability of the feedback system, we first develop a state equation description. Suppose that a minimal realization for $P(s)$ is

$$\begin{aligned} \dot{x}(t)&=Ax(t)+Bu(t)=Ax(t)+B\big(v(t)+w(t)\big)\\ y(t)&=Cx(t)+Du(t)=Cx(t)+D\big(v(t)+w(t)\big) \end{aligned}$$

and a minimal realization for $C(s)$ is

$$\begin{aligned} \dot{z}(t)&=Fz(t)+Ge(t)=Fz(t)+G\big(r(t)+y(t)\big)\\ v(t)&=Hz(t)+Je(t)=Hz(t)+J\big(r(t)+y(t)\big) \end{aligned}$$

It is convenient to compute in partitioned matrix form, so we combine these expressions into the matrix equations

$$\begin{aligned} \begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix}&=\begin{bmatrix}A & 0\\ 0 & F\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}0 & B\\ G & 0\end{bmatrix}\begin{bmatrix}r(t)\\ w(t)\end{bmatrix}+\begin{bmatrix}0 & B\\ G & 0\end{bmatrix}\begin{bmatrix}y(t)\\ v(t)\end{bmatrix}\\ \begin{bmatrix}y(t)\\ v(t)\end{bmatrix}&=\begin{bmatrix}C & 0\\ 0 & H\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}0 & D\\ J & 0\end{bmatrix}\begin{bmatrix}r(t)\\ w(t)\end{bmatrix}+\begin{bmatrix}0 & D\\ J & 0\end{bmatrix}\begin{bmatrix}y(t)\\ v(t)\end{bmatrix} \end{aligned} \tag{12}$$

To solve for the output, write

$$\begin{bmatrix}1 & -D\\ -J & 1\end{bmatrix}\begin{bmatrix}y(t)\\ v(t)\end{bmatrix}=\begin{bmatrix}C & 0\\ 0 & H\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}0 & D\\ J & 0\end{bmatrix}\begin{bmatrix}r(t)\\ w(t)\end{bmatrix}$$

By the assumption that the feedback system is well posed, we have that

$$L \triangleq \begin{bmatrix} 1 & -D \\ -J & 1 \end{bmatrix}$$

is invertible, so that (12) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} &= A_{cl} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + B_{cl} \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \\ \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} &= C_{cl} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + D_{cl} \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{13}$$

where

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} L^{-1} \begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} L^{-1} \begin{bmatrix} 0 & D \\ J & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} L^{-1}(I - L) = \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} L^{-1} \\ C_{cl} &= L^{-1} \begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix} \\ D_{cl} &= L^{-1} \begin{bmatrix} 0 & D \\ J & 0 \end{bmatrix} \end{aligned} \tag{14}$$

Of course, the transfer functions in (7) and (8) are the four entries in the 2×2 transfer function matrix

$$C_{cl}(sI - A_{cl})^{-1} B_{cl} + D_{cl} \tag{15}$$

Next we show, by contradiction arguments, that the closed-loop state equation specified by (13) and (14) is controllable and observable. If it is not controllable, then there exist a scalar λ and nonzero vector $p^T = \begin{bmatrix} p_A^T & p_F^T \end{bmatrix}$ such that

$$p^T A_{cl} = \lambda p^T \,, \quad p^T B_{cl} = p^T \begin{bmatrix} 0 & B \\ G & 0 \end{bmatrix} L^{-1} = 0$$

This gives

$$\begin{bmatrix} p_A^T & p_F^T \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} p_A^T A & p_F^T F \end{bmatrix} = \begin{bmatrix} \lambda p_A^T & \lambda p_F^T \end{bmatrix}, \quad \begin{bmatrix} p_A^T G & p_F^T B \end{bmatrix} = 0$$

But either p_A or p_F is nonzero, and this implies that the corresponding subsystem state equation is not reachable, which is a contradiction. If the closed-loop state equation is not observable, then there exist a scalar λ and nonzero vector

$$p = \begin{bmatrix} p_A \\ p_F \end{bmatrix}$$

such that

$$A_{cl}\, p = \lambda p \,, \quad C_{cl}\, p = L^{-1} \begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix} p = 0$$

This gives

$$\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} p_A \\ p_F \end{bmatrix} = \begin{bmatrix} Ap_A \\ Fp_F \end{bmatrix} = \begin{bmatrix} \lambda p_A \\ \lambda p_F \end{bmatrix}, \quad \begin{bmatrix} Cp_A \\ Hp_F \end{bmatrix} = 0$$

and, as above, a contradiction with observability of the subsystem state equations.
Now suppose that the closed-loop state equation is internally stable, that is, all eigenvalues of A_{cl} have negative real parts. This implies uniform bounded-input, bounded-output stability of the closed-loop system, in particular from each scalar input to each scalar output. Therefore the transfer functions in (9) have negative-real-part poles.
Finally, suppose the closed-loop state equation is not internally stable. Since the state equation is minimal, the closed-loop system cannot be uniformly bounded-input, bounded-output stable, and thus at least one of the transfer functions in (7), (8) has a pole with non-negative real part. Using the lemma, this implies that at least one of the transfer functions in (9) has a pole with non-negative real part.
□

Corollary
Supposing that $C(s)$ has negative real part poles, the well-posed unity feedback system is internally stable if and only if all poles of

$$\frac{Y(s)}{W(s)} = \frac{P(s)}{1 - P(s)C(s)}$$

have negative real parts.

The proof of this result is left as an exercise. There is a symmetric corollary if $P(s)$ is assumed to have negative-real-part poles, and also

Corollary
Supposing that both $C(s)$ and $P(s)$ have negative real part poles, the well-posed unity feedback system is internally stable if and only if all poles of

$$\frac{E(s)}{R(s)} = \frac{1}{1 - P(s)C(s)}$$

have negative real parts.

Stabilizing Controller Parameterization

The set of all (proper, rational) compensator transfer functions, $C(s)$, that yield an internally stable closed-loop system can be described in reasonably explicit terms, especially for the case where $P(s)$ has negative real part poles. Because of its (almost deceptive) simplicity, we will focus on this case even though it excludes the typical situation where $P(s)$ has one or more poles at $s = 0$.

Theorem
Consider the well-posed unity feedback system where all poles of $P(s)$ have negative real parts. The set of all proper rational transfer functions $C(s)$ for which the closed-loop system is internally stable is given by

$$C(s) = \frac{Q(s)}{1 + P(s)Q(s)}$$

where $Q(s)$ is a proper rational function with negative-real-part poles.

Proof
First suppose $C(s)$ is such that the closed-loop system is internally stable. Then, in particular, the transfer function

$$\frac{C(s)}{1-P(s)C(s)}$$

has negative-real-part poles. Let

$$Q(s)=\frac{C(s)}{1-P(s)C(s)}$$

for then elementary algebra gives

$$C(s)=\frac{Q(s)}{1+P(s)Q(s)} \tag{16}$$

and thus $C(s)$ is a member of the claimed set.
Now suppose that $Q(s)$ is a proper rational function with negative-real-part poles. Let $C(s)$ be defined by (16). Then the transfer functions

$$\frac{C(s)}{1-P(s)C(s)}=\frac{\frac{Q(s)}{1+P(s)Q(s)}}{1-\frac{P(s)Q(s)}{1+P(s)Q(s)}}=Q(s)$$

and

$$\frac{P(s)}{1-P(s)C(s)}=\frac{P(s)}{1-\frac{P(s)Q(s)}{1+P(s)Q(s)}}=P(s)+P^2(s)Q(s)$$

both are proper rational with all poles having negative real parts. (It should be clear that sums and products of such transfer functions retain the properties.) Thus by the theorem, the closed-loop system is internally stable.
□

Exercises

1. Can the state equation

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}0 & 1\end{bmatrix}x(t)$$

be made AS by output feedback? Is the state equation reachable? Observable?

2. Show that if

$$rank\begin{bmatrix}\lambda I-A & B\end{bmatrix}=n$$

for all λ with nonnegative real part, then

$$rank\begin{bmatrix}\lambda I-P^{-1}AP & P^{-1}B\end{bmatrix}=n$$

for all λ with nonnegative real part.

3. For the open-loop system described by the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} x(t)$$

is there a linear state feedback

$$u(t) = Kx(t) + Nr(t)$$

such that the closed-loop system has transfer function

$$G_{cl}(s) = \frac{1}{s+3}$$

If so, is the closed-loop system uniformly bounded-input, bounded-output stable? Is it internally stable?

4. Suppose that the linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable and $A + A^T$ is negative semi-definite. Show that the state feedback

$$u(t) = -B^T x(t)$$

yields an asymptotically stable closed-loop state equation. (*Hint*: One approach is to directly consider an arbitrary eigenvalue-eigenvector pair for $A - BB^T$.)

5. *(a)* Consider the linear state equation

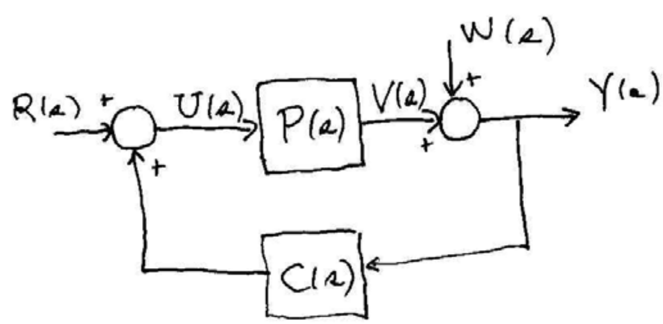
$$\dot{x}(t) = Ax(t) + Bu(t)$$

and suppose the $n \times n$ matrix F has the characteristic polynomial $\det(\lambda I - F) = p(\lambda)$. If the $m \times n$ matrix R and the invertible, $n \times n$ matrix Q are such that

$$AQ - QF = BR$$

show how to choose an $m \times n$ matrix K such that $A + BK$ has characteristic polynomial $p(\lambda)$.
(b) Note that controllability of the open-loop state equation is not assumed. If it is not controllable, what are the implications for this approach to eigenvalue assignment?

6. Consider the feedback system shown below, where $P(s)$ and $C(s)$ are proper rational functions,



and where the system is assumed to be well posed. Determine which two transfer functions must checked to guarantee internal stability.

10. Observers and Output Feedback

The authority of state feedback to adjust the dynamics of an open-loop state equation, assuming controllability, motivates the notion of estimating the state of open-loop system from the output and then using feedback of the state estimate. In addition, the problem of estimating the state from the output is of basic interest on its own.

Full State Observers

Consider the dimension-n linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where the initial state, x_o, is unknown. The objective is to use knowledge of the input and output signals to generate an n-dimensional vector function $\hat{x}(t)$ that is an asymptotic estimate of $x(t)$ in the sense that

$$\lim_{t\to\infty}\left[x(t) - \hat{x}(t)\right] = 0$$

It is natural to think of generating the estimate via another n-dimensional linear state equation that accepts as inputs the signals $u(t)$ and $y(t)$ from (1),

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + Hy(t), \quad \hat{x}(0) = \hat{x}_o \tag{2}$$

where the coefficient matrices F, G, and H, and the initial state $\hat{x}_o$, remain to be specified. Then the estimate error satisfies

$$\dot{x}(t) - \dot{\hat{x}}(t) = (A - HC)x(t) - F\hat{x}(t) + (B - G - HD)u(t), \quad x(0) - \hat{x}(0) = x_o - \hat{x}_o \tag{3}$$

If it happens that $x_o = \hat{x}_o$, a natural requirement to impose on (3) is that $\hat{x}(t) = x(t)$ for all $t \geq 0$. That is, if we actually know the unknown initial state, then the estimate should be exact for all $t \geq 0$. This will be satisfied if we choose

$$F = A - HC, \quad G = B - HD$$

(leaving H unspecified, at this point), for then

$$\dot{x}(t) - \dot{\hat{x}}(t) = [A - HC][x(t) - \hat{x}(t)], \quad x(0) - \hat{x}(0) = 0$$

and, indeed, this implies that $\hat{x}(t) = x(t)$ for all $t \geq 0$.

The choice of coefficients for (3) yields the so-called *state observer*

$$\dot{\hat{x}}(t) = (A - HC)\hat{x}(t) + (B - HD)u(t) + Hy(t), \quad \hat{x}(0) = \hat{x}_o \tag{4}$$

The initial state in (4) might be chosen as a guess at x_o, or it might simply be set to zero.

Defining the state-estimate error as

$$e(t) = x(t) - \hat{x}(t)$$

it follows from (1) and (4) that this error signal satisfies

$$\dot{e}(t) = (A - HC)e(t), \quad e(0) = x_o - \hat{x}_o \tag{5}$$

The state observer will provide an asymptotic estimate of the state if and only if H can be chosen so that (5) is asymptotically stable. A sufficient condition is provided by the following eigenvalue assignability result, which shows that the convergence of the error can be controlled in a quite arbitrary fashion by choice of observer gain.

Theorem
Suppose that the linear state equation (1) is observable. Then given any set of complex numbers, $\lambda_1, \ldots, \lambda_n$, conjugates included, there exists an observer gain H for (4) such that these are the eigenvalues of $A - HC$.

Proof
Since (1) is observable, from Section 5 we can compute an invertible $n \times n$ matrix Q to obtain the observability form coefficients

$$Q^{-1}AQ = \begin{bmatrix} 0 & & & -a_0 \\ 1 & & & \\ & \ddots & & \vdots \\ & & 0 & \\ & & 1 & -a_{n-1} \end{bmatrix}, \quad CQ = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

where the last column of $Q^{-1}AQ$ displays the coefficients of the characteristic polynomial of A. Given a desired set of eigenvalues, $\lambda_1, \ldots, \lambda_n$, let

$$\begin{aligned} p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 \end{aligned}$$

Setting

$$H_{of} = \begin{bmatrix} -p_0 + a_0 & -p_1 + a_1 & \cdots & -p_{n-1} + a_{n-1} \end{bmatrix}$$

an easy calculation gives

$$Q^{-1}AQ - H_{of}CQ = \begin{bmatrix} 0 & & & -p_0 \\ 1 & & & \\ & \ddots & & \vdots \\ & & 0 & \\ & & 1 & -p_{n-1} \end{bmatrix}$$

which shows that observability form is preserved. Moreover, the coefficients of the last column show that this matrix has the desired eigenvalues. Since

$$Q^{-1}AQ - H_{of}CQ = Q^{-1}\left(A - QH_{of}C\right)Q$$

and a similarity transformation does not alter eigenvalues, it is clear that the gain $H = QH_{of}$ is such that $A - HC$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.
□

While observability is a sufficient condition for obtaining an asymptotic estimate of the state, a necessary and sufficient condition can be given as follows.

Theorem
For the linear state equation (1), there exists an observer gain H for (4) such that all eigenvalues of $A - HC$ have negative real parts if and only if

$$\operatorname{rank}\begin{bmatrix} C \\ A - \lambda I \end{bmatrix} = n \tag{6}$$

for every λ that is a negative real part eigenvalue of A.

Proof
From results in Section 5, if

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n\text{-}1} \end{bmatrix} = q$$

for $0 < q < n$, then we can assume that a state variable change has been performed such that

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{7}$$

where A_{11} is $q \times q$, C_1 is $1 \times q$, and

$$rank\begin{bmatrix} C_1 \\ C_1 A_{11} \\ \vdots \\ C_1 A_{11}^{q-1} \end{bmatrix} = q$$

Equivalently,

$$rank\begin{bmatrix} H_1 \\ \lambda I - F_{11} \end{bmatrix} = q$$

for all complex values of λ. Therefore, for any complex λ,

$$rank\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = rank\begin{bmatrix} C_1 & 0 \\ \lambda I - A_{11} & 0 \\ -A_{21} & \lambda I - A_{22} \end{bmatrix} = q + rank\begin{bmatrix} \lambda I - A_{22} \end{bmatrix} \tag{8}$$

If (6) holds for nonnegative-real-part eigenvalues of A, then all eigenvalues of A_{22} have negative real parts. Also there exists H_1 such that $A_{11} - H_1 C_1$ has (desired) eigenvalues, all with negative real parts. Then with

$$H = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}$$

we have that all eigenvalues of

$$A - HC = \begin{bmatrix} A_{11} - H_1 C_1 & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{9}$$

have negative real parts.
On the other hand, suppose H is such that (9) has negative-real-part eigenvalues. Then A_{22} has negative-real-part eigenvalues, and (8) implies that for every λ with nonnegative real part,

$$rank\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} == q + n - q = n$$

□

The rank condition in the theorem is called the *detectability rank condition*.

Reduced-Dimension Observers

The state estimator developed above apparently includes some redundancy because it ignores information provided by the known output signal, $y(t)$. (Of course, this is most clear when the output is one of the state variables – why estimate what is measured?) We next develop an $(n-1)$-dimensional observer that in conjunction with the output signal yields an asymptotic estimate of the state. Because the observer development is a bit more complex, we make the simplifying assumption that $D=0$ in (1).

The first step is to perform a state variable change that clearly delineates the information known about the state. Given the linear state equation (1) with the assumptions that $C \neq 0$ and $D=0$, consider the state variable change

$$z(t) = P^{-1}x(t) = \begin{bmatrix} C \\ P_b \end{bmatrix} x(t)$$

where P_b is arbitrary so long as invertibility holds. This gives, writing $z(t)$ in terms of 1×1 and $(n-1)\times 1$ partitions,

$$\begin{aligned} \begin{bmatrix} \dot{z}_a(t) \\ \dot{z}_b(t) \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u(t) \\ y(t) &= C \begin{bmatrix} C \\ P_b \end{bmatrix}^{-1} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} \end{aligned} \tag{10}$$

Clearly, $z_a(t)$ is given directly by the output signal, but $z_b(t)$ must be estimated. Once this estimate, $\hat{z}_b(t)$, is generated, so that regardless of the initial state of (1) and the initial state of the observer,

$$\lim_{t\to\infty} \left[z_b(t) - \hat{z}_b(t) \right] = 0$$

then an asymptotic estimate of $x(t)$ is given by

$$\hat{x}(t) = \begin{bmatrix} C \\ P_b \end{bmatrix}^{-1} \begin{bmatrix} z_a(t) \\ \hat{z}_b(t) \end{bmatrix} = \begin{bmatrix} C \\ P_b \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \hat{z}_b(t) \end{bmatrix} \tag{11}$$

To generate the asymptotic estimate $\hat{z}_b(t)$, we use an $(n-1)$-dimensional observer of a slightly different form than the full-dimension observer. Specifically, let

$$\begin{aligned} \dot{z}_c(t) &= \tilde{F}z_c(t) + \tilde{G}_a u(t) + \tilde{G}_b y(t) \\ \hat{z}_b(t) &= z_c(t) + Hy(t) \end{aligned} \tag{12}$$

where the coefficients remain to be determined. The estimate error for $z_b(t)$,

$$e_b(t) = z_b(t) - \hat{z}_b(t)$$

satisfies

$$\begin{aligned} \dot{e}_b(t) = \dot{z}_b(t) - \dot{\hat{z}}_b(t) &= \dot{z}_b(t) - \dot{z}_c(t) - H\dot{z}_a(t) \\ &= F_{21}z_a(t) + F_{22}z_b(t) + G_2u(t) - \tilde{F}z_c(t) - \tilde{G}_a u(t) \\ &\quad -\tilde{G}_b z_a(t) - HF_{11}z_a(t) - HF_{12}z_b(t) - HG_1u(t) \end{aligned}$$

Substituting for $z_c(t)$ from the “output” equation in (12) and collecting terms yields

$$\dot{e}_b(t) = \tilde{F}e_b(t) + \left[F_{22} - HF_{12} - \tilde{F}\right]z_b(t) + \left[F_{21} + \tilde{F}H - \tilde{G}_b - HF_{11}\right]z_a(t)$$
$$+\left[G_2 - \tilde{G}_a - HG_1\right]u(t)$$

Now the more-or-less obvious coefficient choices

$$\begin{aligned} \tilde{F} &= F_{22} - HF_{12} \\ \tilde{G}_b &= F_{21} + \left[F_{22} - HF_{12}\right]H - HF_{11} \\ \tilde{G}_a &= G_2 - HG_1 \end{aligned} \tag{13}$$

lead to the error equation

$$\dot{e}_b(t) = \left[F_{22} - HF_{12}\right]e_b(t) \tag{14}$$

and we have the possibility of choosing the $(n-1)\times 1$ gain H to achieve an asymptotic estimate for $\hat{z}_b(t)$. Before addressing this, note that the coefficient choices in (13) specify the observer in (12) as

$$\begin{aligned} \dot{z}_c(t) &= \left[F_{22} - HF_{12}\right]z_c(t) + \left[F_{21} + F_{22}H - HF_{12}H - HF_{11}\right]y(t) \\ &\quad + \left[G_2 - HG_1\right]u(t) \\ \hat{z}_b(t) &= z_c(t) + Hy(t) \end{aligned} \tag{15}$$

where we have written $z_a(t)$ as $y(t)$.

Again we show that if observability is assumed, then the observer gain can be chosen such that the error equation (14) has any desired set of eigenvalues. Thus the rate of convergence of the asymptotic estimate in (11) to the actual state can be arbitrarily set by choice of H .

Theorem
Suppose the linear state equation (1) is observable, and $D = 0$. Then given any set of complex numbers, $\lambda_1, \ldots, \lambda_{n-1}$, conjugates included, there exists an observer gain H such that these are the eigenvalues of $F_{22} - HF_{12}$.

Proof
Observability of (1) implies observability of (10), and we need only show that this implies observability of the $(n-1)$-dimensional state equation

$$\begin{aligned} \dot{z}_d(t) &= F_{22}z_d(t) \\ v(t) &= F_{12}z_d(t) \end{aligned} \tag{16}$$

Proceeding by contradiction, suppose (16) is not observable. Then there exists a nonzero, $(n-1)\times 1$ vector l and a scalar η such that

$$F_{22}l = \eta l\ , \quad F_{12}l = 0$$

But then

$$\begin{bmatrix} 0 \\ l \end{bmatrix}$$

is a nonzero, $n \times 1$ vector such that, using the coefficients of (10),

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} 0 \\ l \end{bmatrix} = \begin{bmatrix} F_{12}l \\ F_{22}l \end{bmatrix} = \eta\begin{bmatrix} 0 \\ l \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ l \end{bmatrix} = 0$$

This implies that (10) is not observable, which in turn implies that (1) is not observable – a contradiction.
□

Remark
If more information about the state can be measured directly, for example, the output is a $p \times 1$ vector signal with $\operatorname{rank} C = p$, then a similar development leads to an observer of dimension $n - p$. □

Observer State Feedback

The notion of using an asymptotic estimate of the state for (linear) feedback yields a powerful approach to considering dynamic output feedback in a state equation setting. We consider the full-dimension observer here, leaving the similar development for the reduced-dimension observer to an exercise.

For the open-loop state equation (1), consider linear dynamic feedback comprising static linear feedback of the state observation provided by

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - HC)\hat{x}(t) + (B - HD)u(t) + Hy(t) \\ u(t) &= K\hat{x}(t) + Nr(t) \end{aligned}$$

Writing the observer as

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - HC)\hat{x}(t) + (B - HD)u(t) + H\left(Cx(t) + Du(t)\right) \\ &= (A - HC)\hat{x}(t) + Bu(t) + HCx(t) \end{aligned}$$

the $2n$-dimensional closed-loop state equation, written in the partitioned form,

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} &= \begin{bmatrix} A & BK \\ HC & A + BK - HC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} BN \\ BN \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & DK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + DNr(t) \end{aligned} \tag{17}$$

Theorem
Suppose the open-loop state equation is controllable and observable. Then the $2n$ eigenvalues of closed-loop state equation (17) are given by the union of the n eigenvalues of $A + BK$ and the n eigenvalues of $A - HC$, both sets that can be arbitrarily assigned by choice of K and H, respectively. Furthermore the closed-loop transfer function is given by

$$\frac{Y(s)}{R(s)} = (C + DK)\left(sI - A - BK\right)^{-1} BN + DN$$

so that the closed-loop poles are independent of the observer.

Proof
To investigate the properties of this state equation, it is convenient to perform the state variable change

$$\begin{bmatrix} z_a(t) \\ z_a(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

where the $2n \times 2n$ variable change matrix is clearly invertible, and in addition is its own inverse. This yields the closed-loop state equation

$$\begin{bmatrix} \dot{z}_a(t) \\ \dot{z}_b(t) \end{bmatrix} = \begin{bmatrix} A+BK & BK \\ 0 & A-HC \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + \begin{bmatrix} BN \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C+DK & -DK \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + DNr(t) \tag{18}$$

From the block-triangular form of the closed-loop A-matrix, the eigenvalue claim follows from previous results. Furthermore, the inverse of an invertible, block-triangular matrix is block triangular, and it turns out that in computing the transfer function for (18) only the diagonal blocks of the inverse are needed. Thus it is straightforward to verify the transfer function claim. □

The rather surprising fact that the eigenvalues of the closed-loop state equation (17) comprise the eigenvalues of $A+BK$ and those of $A-HC$ is called the *separation property* of observer state feedback.

Exercises

*1***.** Suppose the dimension-*n* state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is reachable and observable. Consider the dimension-*2n* state equation comprising this state equation together with its full-dimension observer,

$$\dot{z}(t) = (A-HC)z(t) + HCx(t) + Bu(t)$$

What are the reachability and observability properties of this new state equation?

*2***.** For the plant

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

compute a 2-dimensional observer such that the error decays exponentially with time constants 1/10.

*3***.** Repeat Exercise 2 using a 1-dimensional observer.

*4***.** Suppose the linear state equation

$$\dot{z}(t) = Az(t) + Bu(t)$$
$$y(t) = \begin{bmatrix} I_p & 0 \end{bmatrix} z(t)$$

is reachable and observable. Consider dynamic output feedback of the form

$$u(t) = K\hat{z}(t) + Nr(t)$$

where $\hat{z}(t)$ is a state estimate generated by the reduced-dimension observer discussed in class. Describe the eigenvalues of the closed-loop state equation. What is the closed-loop transfer function?

*5***.** For the linear state equation (*p*-output)

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_o$$
$$y(t) = Cx(t)$$

suppose the $(n-p)\times n$ matrix P_b and the asymptotically stable $(n-p)-$dimensional state equation

$$\dot{z}(t) = \tilde{F}z(t) + \tilde{G}_a u(t) + \tilde{G}_b y(t)$$

satisfy the following additional conditions:

$$rank\begin{bmatrix} C \\ P_b \end{bmatrix} = n\,, \quad 0 = \tilde{F}P_b - P_b A + \tilde{G}_b C\,, \quad \tilde{G}_a = P_b B$$

(a) Show that the $(n-p)\times 1$ error vector $e_b(t) = z(t) - P_b x(t)$ satisfies

$$\dot{e}_b(t) = \tilde{F}e_b(t)$$

(b) Writing

$$\begin{bmatrix} C \\ P_b \end{bmatrix}^{-1} = \begin{bmatrix} H & J \end{bmatrix}$$

where *H* is $n\times p$, show that

$$\hat{x}(t) = Hy(t) + Jz(t)$$

provides an asymptotic estimate for $x(t)$.

*6***.** Suppose that the SISO linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is reachable and observable. Given an $(n-1)\times(n-1)$ matrix *F* and an $n\times 1$ matrix *H*, consider the dynamic output feedback

$$\dot{z}(t) = Fz(t) + Gv(t)$$
$$v(t) = y(t) + CLz(t)$$
$$u(t) = Mz(t) + Nv(t)$$

where the matrices *G, L, M* satisfy

$$AL - BM = LF$$
$$LG + BN = -H$$

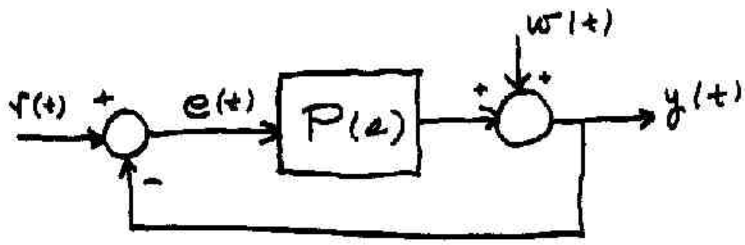
Show that the $2n-1$ eigenvalues of the closed-loop state equation are given by the eigenvalues of *F* and the eigenvalues of $A - HC$. *Hint*: Consider the state variable change defined by

$$\begin{bmatrix} I & L \\ 0 & I \end{bmatrix}$$

11. Output Regulation

As a further illustration of the power of the observer idea in feedback control problems, we consider a special type of reference tracking and disturbance rejection problem that corresponds to an elementary problem in introductory control systems courses.

Example
A familiar example from classical control is the unity-feedback servomechanism shown below



where

$$P(s) = \frac{K}{s(s+a)}$$

with $K, a > 0$. This system has the capability to asymptotically track constant reference inputs, $r(t)$, while asymptotically rejecting constant disturbances. To confirm this, compute the closed-loop transfer functions to obtain

$$\begin{aligned} Y(s) &= \frac{G(s)}{1+G(s)} R(s) + \frac{1}{1+G(s)} W(s) \\ &= \frac{K}{s^2+as+K} R(s) + \frac{s^2+as}{s^2+as+K} W(s) \end{aligned}$$

The closed-loop poles of each transfer function have negative real parts, because of the positive coefficients, and if

$$R(s) = \frac{r_o}{s}, \quad W(s) = \frac{w_o}{s}$$

then, regardless of the values of r_o and w_o ,

$$\begin{aligned} \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) &= \lim_{s\to 0} s \left[\frac{Kr_o}{s(s^2+as+K)} + \frac{(s+a)w_o}{s^2+as+K} \right] \\ &= r_o \end{aligned}$$

□

Problems of the type giving rise to this example can be formulated in terms of state equations as follows. Consider the open-loop state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ y(t) &= Cx(t) + Fw(t) \end{aligned} \tag{1}$$

where $w(t)$ is a scalar disturbance input, and where we have assumed $D = 0$ to de-clutter the development. Using output feedback the objectives for the closed-loop state equation are that the output signal should asymptotically track any constant reference input regardless of any (unknown) constant disturbance input. This is an example of an *output regulation problem*, or *servomechanism problem*.

A key step in the solution of this problem is to assume that the unknown, constant disturbance is provided by a known *exosystem* with unknown initial state, namely

$$\dot{w}(t) = 0\,, \quad w(0) = w_o \tag{2}$$

Then an observer can be used to estimate the state of the augmented state equation comprising the open-loop system and the exosystem:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{3}$$

The full-state observer structure for this augmented open-loop state equation has the form

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{bmatrix} &= \left(\begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right) \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} y(t) \\ &= \begin{bmatrix} A - H_1 C & E - H_1 F \\ -H_2 C & -H_2 F \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} y(t) \end{aligned} \tag{4}$$

Linear feedback of the augmented-state has the form

$$u(t) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + Nr(t) \tag{5}$$

and the resulting closed-loop state equation can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{bmatrix} &= \begin{bmatrix} A & BK_1 & BK_2 \\ H_1 C & A + BK_1 - H_1 C & E + BK_2 - H_1 F \\ H_2 C & -H_2 C & -H_2 F \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} BN \\ BN \\ 0 \end{bmatrix} r(t) + \begin{bmatrix} E \\ H_1 F \\ H_2 F \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + Fw(t) \end{aligned} \tag{6}$$

It remains only to collect the assumptions required to guarantee that the various gains can be chosen to render (6) asymptotically stable with the desired asymptotic output response to constant inputs. The first part is nothing more than an application of full-state observer developments from Section 10 to a (slightly cluttered) augmented state equation.

Theorem

Suppose the linear state equation (1) is controllable (for $E = 0$), the augmented state equation (3) is observable, and the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \tag{7}$$

is invertible. Then given any set of complex numbers, $\lambda_1, \ldots, \lambda_{2n+1}$, conjugates included, there exist gains K_1, H_1 and H_2 such that these are the eigenvalues of the closed-loop state equation (6). Furthermore, assuming these eigenvalues have negative real parts, the gains

$$\begin{aligned} N &= \frac{-1}{C(A + BK_1)^{-1} B} \\ K_2 &= NC(A + BK_1)^{-1} E - NF \end{aligned} \tag{8}$$

are such that for any constant inputs $w(t) = w_o$ and $r(t) = r_o$, $t \ge 0$, the response of the closed-loop state equation has final value r_o.

Proof
The closed-loop state equation (6) is more conveniently analyzed after application of the state variable change

$$z(t) = \begin{bmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{w}(t) \end{bmatrix}$$

where the $(2n+1)\times(2n+1)$ variable change matrix clearly is invertible, and in addition is its own inverse. This gives the closed-loop state equation

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} A+BK_1 & -BK_1 & -BK_2 \\ 0 & A-H_1C & E-H_1F \\ 0 & -H_2C & -H_2F \end{bmatrix} z(t) + \begin{bmatrix} BN \\ 0 \\ 0 \end{bmatrix} r(t) + \begin{bmatrix} E \\ E-H_1F \\ -H_2F \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} C & 0 & 0 \end{bmatrix} z(t) + Fw(t) \end{aligned} \tag{9}$$

Using the block triangular form of the A-matrix in (9), the eigenvalues of the closed-loop state equation are the union of the eigenvalues of

$$A + BK_1$$

and the eigenvalues of

$$\begin{bmatrix} A-H_1C & E-H_1F \\ -H_2C & -H_2F \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix}$$

The controllability and observability hypotheses imply that these eigenvalues can be placed as desired.

To verify the input-output behavior, we compute the closed-loop transfer function in terms of the state equation (9). Careful partitioned calculations verify that

$$\begin{aligned} &\left(sI - \begin{bmatrix} A+BK_1 & -BK_1 & -BK_2 \\ 0 & A-H_1C & E-H_1F \\ 0 & -H_2C & -H_2F \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} (sI-A-BK_1)^{-1} & -(sI-A-BK_1)^{-1}\begin{bmatrix} BK_1 & BK_2 \end{bmatrix} \begin{bmatrix} sI-A+H_1C & -E+H_1F \\ H_2C & sI+H_2F \end{bmatrix}^{-1} \\ 0 & \begin{bmatrix} sI-A+H_1C & -E+H_1F \\ H_2C & sI+H_2F \end{bmatrix}^{-1} \end{bmatrix} \end{aligned}$$

and then partitioned multiplications give

$$\begin{aligned} Y(s) &= C(sI-A-BK_1)^{-1}BN\,R(s) + C(sI-A-BK_1)^{-1}E\,W(s) \\ &\quad - \begin{bmatrix} C(sI-A-BK_1)^{-1}BK_1 & C(sI-A-BK_1)^{-1}BK_2 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} sI-A+H_1C & -E+H_1F \\ H_2C & sI+H_2F \end{bmatrix}^{-1} \begin{bmatrix} E-H_1F \\ -H_2F \end{bmatrix} W(s) + F\,W(s) \end{aligned} \tag{10}$$

Assuming all eigenvalues, and thus poles, of the closed-loop state equation have negative real parts, the final value theorem can be used to compute the final value of the response to the constant input signals,

$$R(s)=\frac{r_o}{s}\,,\quad W(s)=\frac{w_o}{s}$$

From (10), simplifying the most complex term at $s=0$ using

$$\begin{bmatrix} -A+H_1C & -E+H_1F \\ H_2C & H_2F \end{bmatrix}\begin{bmatrix} E-H_1F \\ -H_2F \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

we have

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\
&= \lim_{s\to 0}\left[C(sI-A-BK_1)^{-1}BN\right]r_o \\
&\quad +\lim_{s\to 0}\Big(C(sI-A-BK_1)^{-1}E-\left[C(sI-A-BK_1)^{-1}BK_1 \quad C(sI-A-BK_1)^{-1}BK_2\right] \\
&\qquad \cdot\begin{bmatrix} sI-A+H_1C & -E+H_1F \\ H_2C & sI+H_2F \end{bmatrix}^{-1}\begin{bmatrix} E-H_1F \\ -H_2F \end{bmatrix}\Big)w_o+Fw_o \\
&= -C(A+BK_1)^{-1}BN\, r_o+\left[-C(A+BK_1)^{-1}E-C(A+BK_1)^{-1}BK_2+F\right]w_o
\end{aligned}$$

Now the choices of N and K_2 in (8), with invertibility of $C(A+BK_1)^{-1}B$ provided by the invertibility hypothesis on (7) per Exercise 8.3, yield the claimed final value for the output.
□

Remark
The result remains essentially the same for the case where $y(t)$ and $u(t)$ are $m\times 1$, and $w(t)$ is $q\times 1$, in (1). Also the theory can be generalized significantly to handle asymptotic tracking with bounded, time-varying disturbance inputs generated by LTI exosystems
more general than (2).

Exercises

12. Noninteracting Control

In addition to the powerful properties of state feedback in modifying the dynamics of a given open-loop state equation, state feedback can also modify the input-output structure of multiple-input, multiple-output state equations. In particular, state feedback can be used to isolate selected outputs from selected inputs. We illustrate this capability by considering a very basic problem called, variously, the *noninteracting control* problem, or the *decoupling* problem.

Consider the linear state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \quad x(0) = 0 \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where the input $u(t)$ and the output $y(t)$ both are $m \times 1$ vector signals, $2 \le m \le n$. The objective is to use linear state feedback

$$u(t) = Kx(t) + Nr(t) \tag{2}$$

where $r(t)$ is $m \times 1$ and N is invertible so that the zero-state response of the closed-loop state equation

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + BNr(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

satisfies the following *noninteracting* property. For $i \ne j$ the j^{th}-input component, $r_j(t)$, should have no effect on the i^{th}-output component, $y_i(t)$, for all $t \ge 0$. Put another way, the closed-loop transfer function and/or the closed-loop unit-impulse response should be a diagonal ($m \times m$) matrix.

Remark
The assumption that the input gain N is invertible avoids trivial solutions, for example $N = 0$. Also, it turns out that if the problem has a solution, this assumption guarantees that the noninteracting closed-loop state equation is output controllable (from each input component to its corresponding output component) in the sense of Exercise xx. □

The analysis of the noninteracting control problem further illustrates some of the intricate calculations involved in feedback control. The problem can be addressed either in terms of transfer functions or in terms of unit-impulse responses, and here we use the latter. It is convenient to write the $m \times n$ matrix C in terms of its rows as

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}$$

and to write the closed-loop unit pulse response as

$$H(t) = Ce^{(A+BK)t}BN = \begin{bmatrix} C_1 e^{(A+BK)t}BN \\ \vdots \\ C_m e^{(A+BK)t}BN \end{bmatrix}, \quad t \ge 0$$

Again, the i^{th} row of $H(t)$ should have all entries zero except for the i^{th} entry, $i = 1, \ldots, m$, and this is to be accomplished by choice of the gains K and (invertible) N. The key is to use the power series representation for the matrix exponential, and consider $H(t)$ row-by-row.

Definition
The linear state equation (1) is said to have relative degree $\kappa_1,\ldots,\kappa_m$ if these positive integers are such that

$$C_i A^j B = 0\,, \quad j = 0,\ldots,\kappa_i - 2$$
$$C_i A^{\kappa_i - 1} B \neq 0$$

for $i = 1,\ldots,m$.

The utility of this definition for the present purposes is

Lemma
If the linear state equation (1) has relative degree $\kappa_1,\ldots,\kappa_m$, then for any K and for $i = 1,\ldots,m$,

$$C_i (A+BK)^j = C_i A^j\,, \quad j = 0,\ldots,\kappa_i - 1 \tag{4}$$

Proof
For $j = 0$, the claim is obvious. For any $j > 0$, $C_i (A+BK)^j$ can be written as $C_i A^j$ plus a sum of terms, each with leading factor of one of the forms $C_i B, C_i AB,\ldots,C_i A^{j-1} B$. Thus (4) follows from the definition of relative degree.
□

Theorem
If the linear state equation (1) has relative degree $\kappa_1,\ldots,\kappa_m$, then there exist gains K and invertible N that achieve noninteracting control if and only if the $m \times m$ matrix

$$\Delta = \begin{bmatrix} C_1 A^{\kappa_1 - 1} B \\ \vdots \\ C_m A^{\kappa_m - 1} B \end{bmatrix} \tag{5}$$

is invertible.

Proof
For any $i = 1,\ldots,m$, the i^{th}-row of the closed-loop unit impulse response can be written as

$$C_i e^{(A+BK)t} BN = \sum_{j=0}^{\infty} C_i (A+BK)^j BN \frac{t^j}{j!}$$

The objective is to choose K and N so that (for every i) this has the form $f_i(t) e_i$, where e_i is the i^{th}-row of the $m \times m$ identity matrix. We proceed in a plodding fashion, working through the terms as the index j increases… with somewhat surprising results!
Fixing i, for $j = 0,\ldots,\kappa_i - 1$,

$$C_i (A+BK)^j BN = C_i A^j BN = \begin{cases} 0, & j = 0,\ldots,\kappa_i - 2 \\ C_i A^{\kappa_i - 1} BN, & j = \kappa_i - 1 \end{cases}$$

Choosing the gain $N = \Delta^{-1}$, so that $C_i A^{\kappa_i - 1} BN = e_i$, we meet the objective for $j = 0,\ldots,\kappa_i - 1$. For $j = \kappa_i$,

$$\begin{aligned} C_i (A+BK)^{\kappa_i} BN &= C_i A^{\kappa_i - 1} (A+BK) B \Delta^{-1} \\ &= \left[C_i A^{\kappa_i} + C_i A^{\kappa_i - 1} BK \right] B \Delta^{-1} \end{aligned} \tag{6}$$

Now choose

$$K = -\Delta^{-1} \begin{bmatrix} C_1 A^{\kappa_1} \\ \vdots \\ C_m A^{\kappa_m} \end{bmatrix}$$

Then

$$C_i A^{\kappa_i - 1} BK = -C_i A^{\kappa_i - 1} B\Delta^{-1} \begin{bmatrix} C_1 A^{\kappa_1} \\ \vdots \\ C_m A^{\kappa_m} \end{bmatrix} = -e_i \begin{bmatrix} C_1 A^{\kappa_1} \\ \vdots \\ C_m A^{\kappa_m} \end{bmatrix} = -C_i A^{\kappa_i}$$

so that

$$C_i (A+BK)^{\kappa_i} = C_i A^{\kappa_i - 1}(A+BK) = C_i A^{\kappa_i} + C_i A^{\kappa_i - 1} BK = 0$$

Using this in (6) gives

$$C_i (A+BK)^{\kappa_i} BN = 0$$

Finally, for any $j \geq \kappa_i + 1$,

$$\begin{aligned} C_i (A+BK)^j BN &= C_i (A+BK)^{\kappa_i} (A+BK)^{j-\kappa_i} BN \\ &= 0 \cdot (A+BK)^{j-\kappa_i} BN = 0 \end{aligned}$$

Therefore, by the choices made for K and N, which were independent of the index i, we have

$$Ce^{(A+BK)t} BN = \begin{bmatrix} \dfrac{t^{\kappa_1 - 1}}{(\kappa_1 - 1)!} & & \\ & \ddots & \\ & & \dfrac{t^{\kappa_m - 1}}{(\kappa_m - 1)!} \end{bmatrix} \tag{7}$$

and the closed-loop state equation is noninteracting.

Given the linear state equation (1), with relative degree $\kappa_1, \ldots, \kappa_m$, suppose that gains K and invertible N are such that noninteracting control is achieved. That is, the closed-loop unit impulse response is diagonal. This can be written row-wise as

$$C_i e^{(A+BK)t} BN = h_i(t) e_i \,, \quad i = 1, \ldots, m$$

Replacing the exponential by its power series representation, this implies

$$\sum_{k=0}^{\infty} C_i (A+BK)^k BN \frac{t^k}{k!} = h_i(t) e_i \,, \quad i = 1, \ldots, m$$

Differentiating $\kappa_i - 1$ times and evaluating at $t = 0$ gives

$$C_i (A+BK)^{\kappa_i - 1} BN = h^{(\kappa_i - 1)}(0) e_i \,, \quad i = 1, \ldots, m$$

and using the definition of relative degree, and of N,

$$C_i A^{\kappa_i - 1} BN = e_i \Delta N = h^{(\kappa_i - 1)}(0) e_i \,, \quad i = 1, \ldots, m$$

Each $h_i^{(\kappa_i - 1)}(0) \neq 0$, otherwise, for some i,

$$0 = e_i \Delta N = C_i A^{\kappa_i - 1} BN$$

Since N is invertible, this contradicts the definition of κ_i. Putting the rows together again, we have

$$\Delta N = \begin{bmatrix} h_1^{(\kappa_1 - 1)}(0) & & \\ & \ddots & \\ & & h_m^{(\kappa_m - 1)}(0) \end{bmatrix}$$

and invertibility of the right side, and of N, implies that Δ is invertible.
□

Remark
From (7), the choice of gains in the proof yields a noninteracting closed-loop-system with transfer function

$$C(sI - A - BK)^{-1} BN = \begin{bmatrix} \frac{1}{s^{\kappa_1}} & & \\ & \ddots & \\ & & \frac{1}{s^{\kappa_m}} \end{bmatrix} \qquad (8)$$

Therefore all poles of the transfer functions from the i^{th} input to i^{th} output are at $s = 0$. When Δ is invertible, there are other choices of gains for noninteracting control that might also accomplish additional objectives for the closed-loop state equation. For example, it should be clear that static, linear feedback applied to each of the single-input, single-output subsystems in (8) can provide bounded-input, bounded-output stability. However asymptotic stability is another matter (unless $\kappa_1 + \cdots + \kappa_m = n$), and our treatment is only the beginning of the story.
□

Exercises

*1***.** For what values of the parameter α can the noninteracting control problem be solved for the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

For those values such that the problem can be solved, compute a state feedback control law that yields a noninteracting closed-loop system.

*2***.** For what values of *b* can the noninteracting control problem be solved for the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ b & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

For those values such that the problem can be solved, compute a state feedback control law that yields a noninteracting closed-loop system.

*3***.** Consider a linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

with $p = m$ and consider linear state feedback

$$u(t) = Kx(t) + Nr(t)$$

where $r(t)$ is $m \times 1$. Present conditions under which there exists such a feedback yielding an asymptotically stable closed-loop system with transfer function $G_{cl}(s)$ for which $G_{cl}(0)$ is diagonal and invertible. These requirements define what is sometimes called an "asymptotically noninteracting" closed-loop system. Explain why the terminology is reasonable.